Homogeneity of the pure state space for separable C^*-algebras

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Abstract

We prove that the pure state space is homogeneous under the action of the automorphism group (or a certain smaller group of approximately inner automorphisms) for a fairly large class of simple separable nuclear C^*-algebras, including the approximately homogeneous C^*-algebras and the class of purely infinite C^*-algebras which has been recently classified by Kirchberg and Phillips. This extends the known results for UHF algebras and AF algebras by Powers and Bratteli.

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1 Introduction

If A is a C^*-algebra, we denote by $S(A)$ the convex set of states of A and by $P(A)$ the set of pure states of A. The automorphism group $\mathrm{Aut}(A)$ of A induces an action ϕ on $S(A)$ by affine homeomorphisms, i.e., if $\alpha \in \mathrm{Aut}(A)$, then $\phi(\alpha)$ sends $f \in S(A)$ to $f\alpha^{-1}$. Note that this action leaves $P(A)$ invariant since $P(A)$ is the set of extreme points of $S(A)$ and hence that it cannot be transitive on $S(A)$ (except for the trivial case $A = \mathbf{C}1$). But from Powers' result [11] for UHF algebras we know that $\mathrm{Aut}(A)$ can act transitively on $P(A)$ for some simple C^*-algebras. See [1] for an immediate extension to AF algebras and [3] for a partial extension to Cuntz algebras.

Note that $\mathrm{Aut}(A)$ has some distinguished subgroups; $\mathrm{Inn}(A)$, $\mathrm{AInn}(A)$, $\overline{\mathrm{Inn}}(A)$. Denoting by $\mathcal{U}(A)$ the unitary group of A (or $A + \mathbf{C}1$ if $A \not\ni 1$), the group $\mathrm{Inn}(A)$ of inner automorphisms is given by $\{\mathrm{Ad}\, u \mid u \in \mathcal{U}(A)\}$ and the group $\overline{\mathrm{Inn}}(A)$ of approximately inner automorphisms is the closure of $\mathrm{Inn}(A)$ in $\mathrm{Aut}(A)$ with the topology of strong convergence. The group $\mathrm{AInn}(A)$ of asymptotically inner automorphisms consists of $\alpha \in \overline{\mathrm{Inn}}(A)$ such that there exists a continuous path $(u_t)_{t\in[0,1)}$ in $\mathcal{U}(A)$ with $\alpha = \lim_{t\to 1} \mathrm{Ad}\, u_t$; in the classification theory of purely infinite simple C^*-algebras [8], $\mathrm{AInn}(A)$ is characterized as the group of automorphisms which have the same KK class as the identity automorphism

(see [9] for a similar characterization in the case of simple unital AT algebras of real rank zero). Note that $\mathrm{Inn}(A) \subset \mathrm{AInn}(A) \subset \overline{\mathrm{Inn}}(A) \subset \mathrm{Aut}(A)$ and that the inclusions are proper in general. In the statements given in the previous paragraph $\mathrm{Aut}(A)$ should be replaced by $\overline{\mathrm{Inn}}(A)$ from the way in which they are proven.

In this paper we aim to prove the following statement for a large class of separable C^*-algebras A: If ω_1 and ω_2 are pure states of A with $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$, then there is an $\alpha \in \overline{\mathrm{Inn}}(A)$ such that $\omega_1 = \omega_2\alpha$. Looking at the proofs closely, even more is true in the cases we will handle; it will follow that one can choose α from $\mathrm{AInn}(A)$. Hence, restricted to the case that the C^*-algebra A is simple, we are to try to prove that $\mathrm{AInn}(A)$ acts on $P(A)$ transitively.

In the subsequent section we will introduce some properties for C^*-algebras and show that these properties imply the homogeneity of the pure state space in the above sense if the C^*-algebras are separable. This section contains the main idea of this paper.

In Sections 3–5 we will prove the above properties (more precisely, Property 2.9 below, the strongest among them) for a large class of C^*-algebras, namely approximately homogeneous C^*-algebras, simple crossed products of AF algebras by $\mathbf{Z}$, and a class of purely infinite C^*-algebras including the class which is classified by Kirchberg and Phillips [8, 10]. Hence the above-mentioned transitivity follows for these C^*-algebras. We might expect that this transitivity holds for all separable nuclear C^*-algebras. In section 6 we show that the transitivity holds also for the group C^*-algebras if the group is a countable discrete amenable group.

In Section 7 we note that even a stronger form of transitivity holds for the above mentioned separable C^*-algebras. One consequence is that given any sequence (π_n), indexed by $\mathbf{Z}$, of distinct points in the set of equivalence classes of irreducible representations there is an asymptotically inner automorphism α which shifts this sequence, i.e., $\pi_n\alpha = \pi_{n+1}$ for all n.

The properties for C^*-algebras introduced in Section 2 seem interesting for their own sake. In Section 8 we will give yet another version, which is valid for the C^*-algebras treated above, to derive a closure property saying that if the property holds for a C^*-algebra, then it also holds for all its hereditary C^*-subalgebras. See 8.7 and 8.8 for other consequences.

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2 Homogeneity of the pure state space

Let A be a C^*-algebra and $\mathcal{U}(A)$ the unitary group of A (or $A + \mathbf{C}1$ if $A \not\ni 1$). If two pure states ω_1 and ω_2 are equivalent or $\omega_1 \sim \omega_2$, i.e., the GNS representation π_{ω_1} and π_{ω_2} are equivalent, then there is a $u \in \mathcal{U}(A)$ such that $\omega_1 = \omega_2 \mathrm{Ad}\, u$ by Kadison's transitivity (1.21.16 of [13]); we shall repeatedly use this fact below. First we define the following two properties for A:

Property 2.1 *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If ω_1 and ω_2 are pure states of A such that $\omega_1 \sim \omega_2$ and*

$$|\omega_1(x) - \omega_2(x)| < \delta, \quad x \in \mathcal{G},$$

then there is a $u \in \mathcal{U}(A)$ such that $\omega_1 = \omega_2 \mathrm{Ad}\, u$ and

$$\|\mathrm{Ad}\, u(x) - x\| < \epsilon, \quad x \in \mathcal{F}.$$

Property 2.2 *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If ω_1 and ω_2 are pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$ and*

$$|\omega_1(x) - \omega_2(x)| < \delta, \quad x \in \mathcal{G},$$

then for any finite subset $\mathcal{F}'$ of A and $\epsilon' > 0$ there is a $u \in \mathcal{U}(A)$ such that

$$\begin{aligned} |\omega_1(x) - \omega_2 \mathrm{Ad}\, u(x)| &< \epsilon', \quad x \in \mathcal{F}', \\ \|\mathrm{Ad}\, u(x) - x\| &< \epsilon, \quad x \in \mathcal{F}. \end{aligned}$$

The following lemma is well-known (cf. [4]).

Lemma 2.3 *Let ω_1 and ω_2 be pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$. For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a $u \in \mathcal{U}(A)$ such that*

$$|\omega_1(x) - \omega_2 \mathrm{Ad}\, u(x)| < \epsilon, \quad x \in \mathcal{F}.$$

Proof. Since ω_1 is a pure state, there exists an $e \in A$ such that $e \geq 0$, $\|e\| = 1$, $\omega_1(e) = 1$, and

$$\|exe - \omega_1(x)e^2\| < \epsilon, \quad x \in \mathcal{F}.$$

Since $\|\pi_{\omega_1}(e)\| = 1$, we may suppose, by slightly changing e if necessary, that there is a $\xi \in \mathcal{H}_{\pi_{\omega_2}}$ such that $\|\xi\| = 1$ and $\pi_{\omega_2}(e)\xi = \xi$. Since π_{ω_2} is irreducible there is a $u \in \mathcal{U}(A)$ such that $\pi_{\omega_2}(u^*)\Omega_{\omega_2} = \xi$. Since $\langle \pi_{\omega_2}(exe)\xi, \xi \rangle = \omega_2 \mathrm{Ad}\, u(x)$, we obtain the conclusion. □

Proposition 2.4 *For any C^*-algebra A Property 2.1 implies Property 2.2.*

Proof. Suppose 2.1 and choose $(\mathcal{G}, \delta)$ for $(\mathcal{F}, \epsilon)$ as therein. Suppose that ω_1 and ω_2 are pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$ and

$$|\omega_1(x) - \omega_2(x)| < \delta/2, \quad x \in \mathcal{G}.$$

Let $\mathcal{F}'$ be a finite subset of A and $\epsilon' > 0$ with $\epsilon' < \delta/2$. Then by Lemma 2.3 there is a $v \in \mathcal{U}(A)$ such that

$$|\omega_2 \mathrm{Ad}\, v(x) - \omega_1(x)| < \epsilon', \quad x \in \mathcal{F}' \cup \mathcal{G}.$$

Since $|\omega_2 \mathrm{Ad}\, v(x) - \omega_2(x)| < \delta$, $x \in \mathcal{G}$, there is a $u \in \mathcal{U}(A)$ such that $\|\mathrm{Ad}\, u(x) - x\| < \epsilon$, $x \in \mathcal{F}$ and $\omega_2 \mathrm{Ad}\, v = \omega_2 \mathrm{Ad}\, u$. The latter condition implies that $|\omega_2 \mathrm{Ad}\, u(x) - \omega_1(x)| < \epsilon'$, $x \in \mathcal{F}'$. This completes the proof. □

Theorem 2.5 *Let A be a separable C^*-algebra. Then the following conditions are equivalent:*

1. *Property 2.2 holds.*
2. *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If ω_1 and ω_2 are pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$ and $|\omega_1(x) - \omega_2(x)| < \delta,\ x \in \mathcal{G}$, there is an $\alpha \in \overline{\mathrm{Inn}}(A)$ such that $\omega_1 = \omega_2\alpha$ and $\|\alpha(x) - x\| < \epsilon,\ x \in \mathcal{F}$. If $\mathcal{F} = \emptyset$, then $\mathcal{G}$ can be assumed to be empty.*

Proof. First suppose (2). For any $(\mathcal{F}, \epsilon)$ we choose $(\mathcal{G}, \delta)$ as in (2). Let ω_1 and ω_2 be pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$ and $|\omega_1(x) - \omega_2(x)| < \delta,\ x \in \mathcal{G}$. Then there is an $\alpha \in \overline{\mathrm{Inn}}(A)$ as in (2). Since there is a sequence (u_n) in $\mathcal{U}(A)$ such that $\alpha = \lim \mathrm{Ad}\, u_n$, it follows that for any finite subset $\mathcal{F}'$ of A and $\epsilon' > 0$ there is an n such that $|\omega_1(x) - \omega_2 \mathrm{Ad}\, u_n(x)| < \epsilon',\ x \in \mathcal{F}'$ and $\|\mathrm{Ad}\, u_n(x) - x\| < \epsilon,\ x \in \mathcal{F}$. Thus (1) follows.

Suppose (1). Given $(\mathcal{F}, \epsilon)$ choose $(\mathcal{G}, \delta)$ as in Property 2.2 and let ω_1 and ω_2 be pure states of A such that $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$ and $|\omega_1(x) - \omega_2(x)| < \delta,\ x \in \mathcal{G}$. (If $\mathcal{F} = \emptyset$, then we set $\mathcal{G} = \emptyset$, which will take care of the last statement.)

Let (x_n) be a dense sequence in A.

Let $\mathcal{F}_1 = \mathcal{F} \cup \{x_1\}$ and let $(\mathcal{G}_1, \delta_1)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_1, \epsilon/2)$ as in Property 2.2. By 2.2 there is a $u_1 \in \mathcal{U}(A)$ such that

$$\begin{aligned} \|\mathrm{Ad}\, u_1(x) - x\| &< \epsilon, \quad x \in \mathcal{F}, \\ |\omega_1(x) - \omega_2 \mathrm{Ad}\, u_1(x)| &< \delta_1, \quad x \in \mathcal{G}_1. \end{aligned}$$

Let $\mathcal{F}_2 = \mathcal{F} \cup \{x_1, x_2, \mathrm{Ad}\, u_1^*(x_1), \mathrm{Ad}\, u_1^*(x_2)\}$ and let $(\mathcal{G}_2, \delta_2)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_2, \epsilon/2^2)$ as in Property 2.2 such that $\mathcal{G}_2 \supset \mathcal{G}_1$ and $\delta_2 < \delta_1$. Then there is a $u_2 \in \mathcal{U}(A)$ such that

$$\begin{aligned} \|\mathrm{Ad}\, u_2(x) - x\| &< 2^{-1}\epsilon, \quad x \in \mathcal{F}_1, \\ |\omega_2 \mathrm{Ad}\, u_1(x) - \omega_1 \mathrm{Ad}\, u_2(x)| &< \delta_2, \quad x \in \mathcal{G}_2. \end{aligned}$$

Let $\mathcal{F}_3 = \mathcal{F} \cup \{x_i, \mathrm{Ad}\, u_2^*(x_i) \mid i = 1, 2, 3\}$ and let $(\mathcal{G}_3, \delta_3)$ be the $(\mathcal{G}, \delta)$ for $(\mathcal{F}_3, 2^{-3}\epsilon)$ as in Property 2.2 such that $\mathcal{G}_3 \supset \mathcal{G}_2$ and $\delta_3 < \delta_2$. Then there is a $u_3 \in \mathcal{U}(A)$ such that

$$\begin{aligned} \|\mathrm{Ad}\, u_3(x) - x\| &< 2^{-2}\epsilon, \quad x \in \mathcal{F}_2, \\ |\omega_1 \mathrm{Ad}\, u_2(x) - \omega_2 \mathrm{Ad}\, u_1 u_3(x)| &< \delta_3, \quad x \in \mathcal{G}_3. \end{aligned}$$

We shall repeat this process.

If $\mathcal{F}_k, \mathcal{G}_k, \delta_k$, and u_k are given for $k < n$, let

$$\mathcal{F}_n = \mathcal{F} \cup \{x_i, \mathrm{Ad}\, u_{n-1}^* u_{n-3}^* \cdots u_\#^*(x_i) \mid i = 1, 2, \ldots, n\},$$

where $\# = 2$ or 1 depending on the parity of n. We then choose $(\mathcal{G}_n, \delta_n)$ as in Property 2.2 for $(\mathcal{F}_n, 2^{-n}\epsilon)$ such that $\mathcal{G}_n \supset \mathcal{G}_{n-1}$ and $\delta_n < \delta_{n-1}$. Finally we pick up a $u_n \in \mathcal{U}(A)$ such that

$$\|\mathrm{Ad}\, u_n(x) - x\| < 2^{-n+1}\epsilon, \quad x \in \mathcal{F}_{n-1}$$

and if n is odd,

$$|\omega_1\mathrm{Ad}(u_2u_4\cdots u_{n-1})(x)-\omega_2\mathrm{Ad}(u_1u_3\cdots u_n)(x)|<\delta_n,\quad x\in\mathcal{G}_n,$$

else if n is even,

$$|\omega_1\mathrm{Ad}(u_2u_4\cdots u_n)(x)-\omega_2\mathrm{Ad}(u_1u_3\cdots u_{n-1})(x)|<\delta_n,\quad x\in\mathcal{G}_n.$$

We may assume that $\cup_n\mathcal{G}_n$ is dense in A.

Since $\|\mathrm{Ad}\,u_k(x_i)-x_i\|<2^{-k+1}\epsilon$ for $k>i$, we have that

$$\lim_{n\longrightarrow\infty}\mathrm{Ad}(u_1u_3\cdots u_{2n-1})(x_i)$$

converges for any i. Since (x_i) is a dense sequence in A, we have that $\mathrm{Ad}(u_1u_3\cdots u_{2n-1})$ converges strongly on A; thus the limit α exists as an endomorphism of A. In a similar way $\mathrm{Ad}(u_2u_4\cdots u_{2n})$ converges to an endomorphism β of A. From the estimate on $\omega_1\mathrm{Ad}(u_2u_4\cdots u_{2n})-\omega_2\mathrm{Ad}(u_1u_3\cdots u_{2n+1})$, it follows that $\omega_1\beta=\omega_2\alpha$.

We shall show that α and β are automorphisms. For that purpose it suffices to show that $\mathrm{Ad}(u_{2n}^*u_{2n-2}^*\cdots u_2^*)$ and a similar expression with odd-numbered u_k converge as $n\rightarrow\infty$.

Since $\|\mathrm{Ad}\,u_n(x)-x\|<2^{-n+1}\epsilon,\ x\in\mathcal{F}_{n-1}$, we have that if $n>i$,

$$\|\mathrm{Ad}\,u_n\,\mathrm{Ad}(u_{n-2}^*u_{n-4}^*\cdots u_{\#}^*)(x_i)-\mathrm{Ad}(u_{n-2}^*u_{n-4}^*\cdots u_{\#}^*)(x_i)\|<2^{-n+1}\epsilon.$$

This implies the desired convergence.

Finally, since $\mathcal{F}_n\supset\mathcal{F}$, it follows that the automorphisms $\alpha=\lim\mathrm{Ad}(u_1u_3\cdots u_{2n-1})$ and $\beta=\lim\mathrm{Ad}(u_2u_4\cdots u_{2n})$ satisfy that

$$\|\alpha(x)-x\|<4\epsilon/3,\quad\|\beta(x)-x\|<2\epsilon/3,$$

for $x\in\mathcal{F}$. Hence $\|\alpha\beta^{-1}(x)-x\|<2\epsilon,\ x\in\mathcal{F}$. Since $\omega_1=\omega_2\alpha\beta^{-1}$, this completes the proof. □

In the following sections we will actually treat properties stronger than 2.1.

Property 2.6 *For any finite subset $\mathcal{F}$ of A and $\epsilon>0$ there exist a finite subset $\mathcal{G}$ of A and $\delta>0$ satisfying: If ω_1 and ω_2 are pure states of A such that $\omega_1\sim\omega_2$ and*

$$|\omega_1(x)-\omega_2(x)|<\delta,\quad x\in\mathcal{G},$$

then there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(A)$ such that $u_0=1$, $\omega_1=\omega_2\mathrm{Ad}\,u_1$, and

$$\|\mathrm{Ad}\,u_t(x)-x\|<\epsilon,\quad x\in\mathcal{F},\ t\in[0,1].$$

As in the proof of 2.4 we can show that the above property implies:

Property 2.7 *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If ω_1 and ω_2 are pure states of A such that* $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$ *and*

$$|\omega_1(x) - \omega_2(x)| < \delta, \quad x \in \mathcal{G},$$

then for any finite subset $\mathcal{F}'$ of A and $\epsilon' > 0$ there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(A)$ such that $u_0 = 1$, and

$$\begin{aligned} |\omega_1(x) - \omega_2 \mathrm{Ad}\, u_1(x)| &< \epsilon', \quad x \in \mathcal{F}', \\ \|\mathrm{Ad}\, u_t(x) - x\| &< \epsilon, \quad x \in \mathcal{F}, \ t \in [0,1]. \end{aligned}$$

We recall here that $\mathrm{AInn}(A)$ is the group of asymptotically inner automorphisms of A, a proper normal subgroup of $\overline{\mathrm{Inn}}(A)$ in general. The following result can be shown exactly in the same way as Theorem 2.5 is shown:

Theorem 2.8 *Let A be a separable C^*-algebra. Then the following conditions are equivalent:*

1. *Property 2.7 holds.*
2. *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If ω_1 and ω_2 are pure states of A such that* $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$ *and $|\omega_1(x) - \omega_2(x)| < \delta$, $x \in \mathcal{G}$, there exist an $\alpha \in$* $\mathrm{AInn}(A)$ *and a continuous path $(u_t)_{t\in[0,1)}$ in $\mathcal{U}(A)$ such that $u_0 = 1$, $\alpha = \lim_{t\to 1} \mathrm{Ad}\, u_t$, $\omega_1 = \omega_2 \alpha$ and $\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon$, $x \in \mathcal{F}$, $t \in [0,1)$. If $\mathcal{F} = \emptyset$, then $\mathcal{G}$ can be assumed to be empty.*

Proof. The proof proceeds exactly in the same way as the proof of 2.5. In the proof of (1)⇒(2) of 2.5 we have defined the automorphisms α and β of A; the α in the above statement is $\alpha\beta^{-1}$. The α in the proof of 2.5 is defined as the limit of $\mathrm{Ad}(u_1 u_3 \cdots u_{2n-1})$. In the present assumption we have a continuous path (u_{nt}) in $\mathcal{U}(A)$ for each n such that $u_{n0} = 1$, $u_{n1} = u_n$, and $\|\mathrm{Ad}\, u_{nt}(x) - x\| < 2^{-n+1}\epsilon$, $x \in \mathcal{F}_{n-1}$. We define a continuous path $(v_t)_{t\in[0,\infty)}$ by: for $t \in [n, n+1]$,

$$v_t = u_1 u_3 \cdots u_{2n-1} u_{2n+1,t-n}.$$

Then it follows that $v_0 = 1$, $\alpha = \lim_{t\to\infty} \mathrm{Ad}\, v_t$, and $\|\mathrm{Ad}\, v_t(x) - x\| \le 4\epsilon/3$, $x \in \mathcal{F}$. We prove that β also enjoys a similar property and thus $\alpha\beta^{-1}$ does too. This shows that (1)⇒(2).

The other implication is obvious since it is assumed that $u_0 = 1$ for the path (u_t) in (2). □

We will consider even a stronger property:

Property 2.9 *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If B is a C^*-algebra containing A as a C^*-subalgebra and ω_1 and ω_2 are pure states of B such that $\omega_1 \sim \omega_2$ and*

$$|\omega_1(x) - \omega_2(x)| < \delta, \quad x \in \mathcal{G},$$

then there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(B)$ such that $u_0 = 1$, $\omega_1 = \omega_2 \mathrm{Ad}\, u_1$, and

$$\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F}, \ t \in [0,1].$$

In the above property we may replace the exact equality $\omega_1 = \omega_2 \mathrm{Ad}\, u_1$ by an approximate one $\|\omega_1 - \omega_2 \mathrm{Ad}\, u_1\| < \epsilon$; by Kadison's transitivity one can make a small modification to (u_t) to get the exact equality.

Obviously 2.9⇒2.6⇒2.7 among the properties defined above.

Lemma 2.10 *When A is unital, Property 2.9 is equivalent to the one obtained by restricting the ambient algebra B to a C^*-algebra having 1_A as a unit.*

Proof. Given $(\mathcal{F}, \epsilon)$ let $(\mathcal{G}, \delta)$ be the one obtained in this weaker property. Let $\mathcal{G}_1 = \mathcal{G} \cup \{N1_A\}$, where N will be specified later to be a large positive number, and suppose that B is given such that $B \supset A$ and 1_A is not an identity for B. If π is an irreducible representation of B and ω_1, ω_2 are pure states of B defined by the unit vectors $\xi_1, \xi_2 \in \mathcal{H}_\pi$ respectively such that $|\omega_1(x) - \omega_2(x)| < 2^{-1}\epsilon^2\delta$, $x \in \mathcal{G}_1$, then we can choose N so large that $|\omega_1(1_A) - \omega_2(1_A)|$ is very small and either, $\omega_1(1_A)$ and $\omega_2(1_A)$ are smaller than ϵ^2, or $|\phi_1(x) - \phi_2(x)| < \delta$, $x \in \mathcal{G}$ for the states $\phi_i = \omega_i(1_A)^{-1}\omega_i|1_A B 1_A$. In any case we can choose a continuous path $(w_t)_{t\in[0,1]}$ in $\mathcal{U}((1-1_A)B(1-1_A))$ such that $\|(1-\pi(1_A))\xi_1 - \pi(w_1)\xi_2\|$ is very small. In the former case where $\|\pi(1_A)\xi_i\| < \epsilon$, we set $v_t = 1_A$ and in the latter case we choose, by the weaker property, a continuous path $(v_t)_{t\in[0,1]}$ in $\mathcal{U}(1_A B 1_A)$ such that $\|\pi(1_A)\xi_1 - \pi(v_1)\xi_2\|$ is very small and $\|\mathrm{Ad}\, v_t(x) - x\| < \epsilon$, $x \in \mathcal{F}$. Setting $u_t = w_t + v_t$, it follows that $\|\xi_1 - \pi(u_1)\xi_2\|$ is of order ϵ and $\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon$, $x \in \mathcal{F}$. This completes the proof. □

Proposition 2.11 *If $\mathcal{C}$ denotes the class of C^*-algebras with Property 2.9, then the following statements hold:*

1. *If $A \in \mathcal{C}$ is non-unital, then $\tilde{A} \in \mathcal{C}$, where $\tilde{A}$ is the C^*-algebra obtained by adjoining a unit to A.*
2. *If $A_1, A_2 \in \mathcal{C}$ are unital, then $A_1 \oplus A_2 \in \mathcal{C}$.*
3. *If $A \in \mathcal{C}$ and $e \in A$ is a projection, then $eAe \in \mathcal{C}$.*
4. *If $A \in \mathcal{C}$ and I is an ideal of A, then the quotient $A/I \in \mathcal{C}$.*
5. *If (A_n) is an inductive system with $A_n \in \mathcal{C}$, then $\lim_n A_n \in \mathcal{C}$.*

Proof. To prove (1) we may assume that $1_B = 1_{\tilde{A}}$ in Property 2.9. Then this is obvious.

To prove (2) let $\mathcal{F}$ be a finite subset of $A \equiv A_1 \oplus A_2$ and $\epsilon > 0$. Then there is a finite subset $\mathcal{F}_i \subset A_i$ for each $i = 1, 2$ such that if $\|\mathrm{Ad}\, u(x) - x\| < \epsilon,\ x \in \mathcal{F}_1 \cup \mathcal{F}_2$, then $\|\mathrm{Ad}\, u(x) - x\| < \epsilon,\ x \in \mathcal{F}$, for any $u \in \mathcal{U}(B)$ with $B \supset A$ and $1_B = 1_A$. Choose $(\mathcal{G}_i, \delta_i)$ for $(\mathcal{F}_i, \epsilon)$ as in Property 2.9. We may suppose that $\delta_1 = \delta_2 \equiv \delta$ by multiplying each element of $\mathcal{G}_1$ by δ_2/δ_1. Let $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \{Np_1, Np_2\}$, where p_i is the identity of A_i in B (and so $p_1 + p_2 = 1$). Let π be an irreducible representation of B and let ω_i be a pure state of B defined by a unit vector $\xi_i \in \mathcal{H}_\pi$ for $i = 1, 2$.

By choosing N sufficiently large, we may assume that if $|\omega_1(x) - \omega_2(x)| < \epsilon^2\delta/2,\ x \in \mathcal{G}$, then it follows that $|\omega_1(p_i) - \omega_2(p_i)|$ is very small for $i = 1, 2$, and both $\omega_1(p_i)$ and $\omega_2(p_i)$ are smaller than ϵ^2 or

$$\left|\frac{\omega_1(x)}{\omega_1(p_i)} - \frac{\omega_2(x)}{\omega_2(p_i)}\right| < \delta, \quad x \in \mathcal{G}_i.$$

In the former case we set $u_{it} = p_i$ and in the latter case we obtain a continuous path (u_{it}) in $\mathcal{U}(p_iBp_i)$ such that $u_{i0} = p_i$, and

$$\begin{aligned} \|\mathrm{Ad}\, u_{it}(x) - x\| &< \epsilon, \quad x \in \mathcal{F}_i, \\ \omega_1(p_i)^{-1}\omega_1|p_iBp_i &= \omega_2(p_i)^{-1}\omega_2 \mathrm{Ad}\, u_{i1}|p_iBp_i. \end{aligned}$$

By multiplying (u_{it}) by a $\mathbf{T}$-valued continuous function, we may further suppose that $\pi(u_{i1})\xi_1 = \|\pi(p_i)\xi_1\| \|\pi(p_i)\xi_2\|^{-1}\pi(p_i)\xi_2$, where $\|\pi(p_i)\xi_1\|\|\pi(p_i)\xi_2\|^{-1}$ is arbitrarily close to 1. Set $u_t = u_{1t} + u_{2t}$; then $\|\omega_1 - \omega_2 \mathrm{Ad}\, u_1\|$ is of the order of ϵ (since $\|\pi(u_1)\xi_1 - \xi_2\|$ is at most of order ϵ) and $\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon,\ x \in \mathcal{F}_1 \cup \mathcal{F}_2$.

To prove (3) let e be a projection in $A \in \mathcal{C}$ and let B be a C^*-algebra with $B \supset eAe$ and $1_B = e$. Then there is a C^*-algebra D such that $eDe = B$ and the diagram

$$\begin{array}{ccc} eAe & \subset & B \\ \cap & & \cap \\ A & \subset & D \end{array}$$

is commutative. (To show this we may suppose that A acts on a Hilbert space $\mathcal{H}$ and that B acts on $e\mathcal{H}$ with $B \supset eAe$. We set D to be the C^*-algebra generated by B and A.) Let $\mathcal{F}$ be a finite subset of eAe and $\epsilon > 0$. Let $\mathcal{F}_1 = \mathcal{F} \cup \{Ne\}$ with N sufficiently large. We choose $(\mathcal{G}, \delta)$ for $(\mathcal{F}_1, \epsilon)$ from Property 2.9 of A and set $\mathcal{G}_1 = \{exe \mid x \in \mathcal{G}\}$. Let ω_1, ω_2 be pure states of B such that $\omega_1 \sim \omega_2$ and $|\omega_1(x) - \omega_2(x)| < \delta,\ x \in \mathcal{G}_1$. Extend ω_i to a pure state of D; since the extension is unique, we denote it by the same symbol. Since $\omega_i(x) = \omega_i(exe)$, we obtain, from Property 2.9, a continuous path (u_t) in $\mathcal{U}(D)$ such that $u_0 = 1$, $\omega_1 = \omega_2 \mathrm{Ad}\, u_1$, and $\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon,\ x \in \mathcal{F}_1$. Since $\|[e, u_t]\|$ is very small, we can obtain a continuous path in $\mathcal{U}(eAe)$ from (eu_te) which satisfies the required properties.

To prove (4) let B a C^*-algebra with $B \supset A/I$. Let $D = B \oplus A$ and embed A into D by $x \mapsto (x + I, x)$. Let $\mathcal{F}$ be a finite subset of A/I and $\epsilon > 0$. For each $x \in \mathcal{F}$ choose an $x_1 \in A$ such that $x_1 + I = x$ and denote by $\mathcal{F}_1$ the subset consisting of such x_1 with

$x \in \mathcal{F}$. For $(\mathcal{F}_1, \epsilon)$ we obtain $(\mathcal{G}_1, \delta)$ as in Property 2.9. Denoting by p the projection of D onto B, we set $\mathcal{G} = \{p(x) \mid x \in \mathcal{G}_1\}$. Then it is easy to check that $(\mathcal{G}, \delta)$ satisfies the requires properties for $(\mathcal{F}, \epsilon)$.

To show (5) note, by (4), that (A_n) can be regarded as an increasing sequence in $\mathcal{C}$. Then for any finite subset $\mathcal{F}$ of $\overline{\cup_n A_n}$ we find A_n which almost contains $\mathcal{F}$. Hence this is immediate. □

3 Homogeneous C^*-algebras

In this section we will show that the C^*-algebras of the form $C \otimes M_n$ have Property 2.9, where C is a unital abelian C^*-algebra. Then it will follow by 2.11 that any approximately homogeneous C^*-algebra has Property 2.9. Furthermore we will attempt to prove that some subhomogeneous C^*-algebras have Property 2.9.

The following is well-known.

Lemma 3.1 *For any $\epsilon > 0$ there is a $\delta > 0$ satisfying: If A is a C^*-algebra and $A_{+1} = \{x \in A \mid x \geq 0, \|x\| \leq 1\}$, then that $\|x - y\| < \delta$ implies that $\|x^{1/2} - y^{1/2}\| < \epsilon$ for any $x, y \in A_{+1}$.*

Proof. For any $\epsilon > 0$ there is a real-valued polynomial $p(t)$ with $p(0) = 0$ such that $|t^{1/2} - p(t)| < \epsilon,\ t \in [0, 1]$. If $p(t) = \sum_{i=1}^n a_i t^i$, set $C = \sum_i |a_i| i$. Then for $x, y \in A_{+1}$ we have that $\|p(x) - p(y)\| \leq C\|x - y\|$. Hence for $x, y \in A_{+1}$ the condition that $\|x - y\| < \epsilon/C$ implies that $\|x^{1/2} - y^{1/2}\| < 3\epsilon$. □

Lemma 3.2 *For any $\epsilon > 0$ and $n \in \mathbf{N}$ there exists a $\delta > 0$ satisfying: If $\xi_1, \xi_2, \ldots, \xi_n$ are vectors in a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq n$ and $c = (c_{ij})$ is an $n \times n$ matrix such that $\sum_{i=1}^n \|\xi_i\|^2 \leq 1$, $c \geq 0$, and*

$$|c_{ij} - \langle \xi_i, \xi_j \rangle| < \delta, \quad i, j = 1, 2, \ldots, n,$$

then there exist vectors $\eta_1, \eta_2, \ldots, \eta_n$ in $\mathcal{H}$ such that

$$\begin{aligned} \langle \eta_i, \eta_j \rangle &= c_{ij}, \quad i, j = 1, 2, \ldots, n, \\ \|\xi_i - \eta_i\| &< \epsilon, \quad i = 1, 2, \ldots, n. \end{aligned}$$

Proof. Let $d_{ij} = \langle \xi_i, \xi_j \rangle$. Then the $n \times n$ matrix $d = (d_{ij})$ satisfies that $d \geq 0$ and $\|d\| \leq 1$. The condition that $|c_{ij} - d_{ij}| < \delta$ for all i, j implies that $\|c - d\| \leq n\delta$.

If d is strictly positive, then define $\eta_i = \sum_{k=1}^n (c^{1/2} d^{-1/2})_{ik} \xi_k$. Then by computation

$$\langle \eta_i, \eta_j \rangle = \sum_{k,\ell} (c^{1/2} d^{-1/2})_{ik} \overline{(c^{1/2} d^{-1/2})}_{j\ell} d_{k\ell} = c_{ij}$$

and

$$\|\eta_i - \xi_i\|^2 = (c^{1/2} - d^{1/2})^2_{ii} \le \|c^{1/2} - d^{1/2}\|^2.$$

By the previous lemma one can choose $\delta > 0$ so small that $\|c^{1/2} - d^{1/2}\| < \epsilon$ follows from $\|c - d\| \le n\delta$ and $\|d\| \le 1$.

In general let $\mathcal{L}$ be a subspace of $\mathcal{H}$ such that $\mathcal{L} \ni \xi_i$ for all i and $\dim \mathcal{L} = n$. Then there is a sequence $(\xi_{k1}, \xi_{k2}, \dots, \xi_{kn})$ of bases of $\mathcal{L}$ such that $\|\xi_{ki} - \xi_i\| \to 0$ as $k \to \infty$. For each $(\xi_{k1}, \xi_{k2}, \dots, \xi_{kn})$ one can apply the previous argument to produce vectors $\eta_{k1}, \eta_{k2}, \dots, \eta_{kn}$ in $\mathcal{L}$. By using the compactness argument for the limit of k to infinity one obtains vectors $\eta_1, \eta_2, \dots, \eta_n$ in $\mathcal{L}$ such that $\langle \eta_i, \eta_j \rangle = c_{ij}$ and $\|\eta_i - \xi_i\|^2 = (c^{1/2} - d^{1/2})^2_{ii}$. This completes the proof. □

Lemma 3.3 *For any $\epsilon > 0$ and $n \in \mathbf{N}$ there exists a $\delta > 0$ satisfying: Given sequences $(\xi_1, \xi_2, \dots, \xi_n)$ and $(\eta_1, \eta_2, \dots, \eta_n)$ of vectors in a Hilbert space $\mathcal{H}$ such that $\sum_i \|\xi_i\|^2 \le 1$, $\sum_i \|\eta_i\|^2 \le 1$, and*

$$|\langle \xi_i, \xi_j \rangle - \langle \eta_i, \eta_j \rangle| < \delta, \quad i, j = 1, 2, \dots, n,$$

there is a unitary U on $\mathcal{H}$ such that

$$\|U\xi_i - \eta_i\| < \epsilon, \quad i = 1, 2, \dots, n.$$

Proof. If $\dim \mathcal{H} \ge n$, then choose $\delta > 0$ as in the previous lemma and find a sequence $(\zeta_1, \zeta_2, \dots, \zeta_n)$ in $\mathcal{H}$ such that $\langle \zeta_i, \zeta_j \rangle = \langle \xi_i, \xi_j \rangle$ and $\|\zeta_i - \eta_i\| < \epsilon$. Then we can find a unitary U on $\mathcal{H}$ such that $U\xi_i = \zeta_i$ for $i = 1, 2, \dots, n$, which satisfies the required properties.

If $m = \dim \mathcal{H} < n$, we first choose m vectors from $(\xi_1, \xi_2, \dots, \xi_n)$. Pick up ξ_{i_1} with $\|\xi_{i_1}\| = \max_j \|\xi_j\|$. Let P_1 be the projection onto the subspace spanned by ξ_{i_1} and pick up ξ_{i_2} with $\|(1 - P_1)\xi_{i_2}\| = \max_j \|(1 - P_1)\xi_j\|$. Let P_2 be the projection onto the subspace spanned by ξ_{i_1}, ξ_{i_2} and pick up ξ_{i_3} with $\|(1 - P_2)\xi_{i_3}\| = \max_j \|(1 - P_2)\xi_j\|$. Repeating this process we obtain $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_m}$, which we assume are all different as we may. If $j \notin I = \{i_1, i_2, \dots, i_m\}$, then there are $c_j^k \in \mathbf{C}$ for $k \in I$ such that

$$\xi_j = \sum_{k \in I} c_j^k \xi_k.$$

From the construction of I we can assume that $|c_j^k| \le 1$. (If ξ_k, $k \in I$ are linearly independent, then of course $|c_j^k| \le 1$ automatically.) If $|\langle \xi_i, \xi_j \rangle - \langle \eta_i, \eta_j \rangle| < \delta$ for all i, j, then $\|\eta_j - \sum_{k \in I} c_j^k \eta_k\|^2 \le (m + 1)^2 \delta$. Thus if we define a unitary U on $\mathcal{H}$ by requiring that $\|U\xi_k - \eta_k\| < \epsilon$, $k \in I$, we obtain that for $j \notin I$,

$$\|U\xi_j - \eta_j\| \le \sum_{k \in I} |c_j^k| \|U\xi_k - \eta_k\| + \|\eta_j - \sum_{k \in I} c_j^k \eta_k\| \le m\epsilon + (m + 1)^2 \delta.$$

This completes the proof. □

Lemma 3.4 *Any matrix algebra M_n has Property 2.9.*

Proof. Let B be a C^*-algebra containing $A = M_n$ with $1_B = 1_A$. Let (e_{ij}) be the set of matrix units of $A = M_n$. Since A is spanned by the matrix units, we may take $\{e_{ij} \mid i, j = 1, 2, \ldots, n\}$ for both $\mathcal{F}$ and $\mathcal{G}$ in Property 2.9. Let $\epsilon > 0$ and let $\delta > 0$ be the δ for $\epsilon n^{-1/2}$ in place of ϵ in Lemma 3.3.

Let π be an irreducible representation of B on $\mathcal{H}$ and let ξ, η be unit vectors in $\mathcal{H}$ such that

$$|\langle \pi(e_{ij})\xi, \xi\rangle - \langle \pi(e_{ij})\eta, \eta\rangle| < \delta$$

for all i, j. We apply Lemma 3.3 to the sequences $(\pi(e_{1j})\xi)$ and $(\pi(e_{1j})\eta)$ and obtain a unitary U on the subspace $\mathcal{L}$ spanned by $(\pi(e_{1j})\xi)$ and $(\pi(e_{1j})\eta)$ such that $\|U\pi(e_{1j})\xi - \pi(e_{1j})\eta\| < \epsilon n^{-1/2}$. Since $U = e^{iH}$ with $\|H\| \leq \pi$ we choose a self-adjoint $h = h^* \in e_{11}Be_{11}$ such that $\|h\| \leq \pi$ and $\pi(h)|\mathcal{L} = H$. Define a unitary u_t for each $t \in [0, 1]$ by

$$u_t = \sum_i e_{i1} e^{\sqrt{-1}th} e_{1i}.$$

Then it follows that $u_t \in B \cap A'$. By computation

$$\begin{aligned} \|\pi(u_1)\xi - \eta\|^2 &= \|\sum_i \{\pi(e_{i1})U\pi(e_{1i})\xi - \pi(e_{ii})\eta\}\|^2 \\ &= \sum_i \|U\pi(e_{1i})\xi - \pi(e_{1i})\eta\|^2 \\ &< \epsilon^2. \end{aligned}$$

Thus $\|\pi(u_1)\xi - \eta\| < \epsilon$. Since $u_t \in B \cap A'$, this completes the proof. $\square$

Proposition 3.5 *If A is a unital C^*-algebra satisfying Property 2.9 and C is a unital commutative C^*-algebra, then $A \otimes C$ satisfies Property 2.9*

Since any unital commutative C^*-algebra is an inductive limit of quotients of $C(\mathbf{T}^n)$, we may assume that $C = C(\mathbf{T})$ in the above proposition by 2.11. This follows essentially from the following lemma.

Lemma 3.6 *For any small $\epsilon > 0$ and $\epsilon' > 0$ there exist a $\delta > 0$ and $n \in \mathbf{N}$ satisfying: Let U be a unitary operator on a Hilbert space $\mathcal{H}$ with E its spectral measure and let ξ, η be two unit vectors in $\mathcal{H}$. If*

$$|\langle U^k\xi, \xi\rangle - \langle U^k\eta, \eta\rangle| < \delta$$

for $k = 0, \pm 1, \ldots, \pm n$, there exists a sequence $(t_1, t_2, \ldots, t_m)$ of points in $\mathbf{T}$ such that $t_{i-1} < t_i$ in the cyclic order with $t_0 = t_m$, $\epsilon/2 < \mathrm{dist}(t_{i-1}, t_i) < 3\epsilon/2$ and

$$\begin{aligned} &|\|E(t_{i-1}, t_i)\xi\|^2 - \|E(t_{i-1}, t_i)\eta\|^2| < 2\epsilon', \\ &\|E(t_i - \gamma/2, t_i + \gamma/2)\xi\|^2 < \epsilon', \quad \|E(t_i - \gamma/2, t_i + \gamma/2)\eta\|^2 < \epsilon', \end{aligned}$$

where $\gamma = \epsilon\epsilon'/4$ and the total length of $\mathbf{T}$ is normalized to be 1.

Proof. Let $\gamma = \epsilon\epsilon'/4$. For any interval I of $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ of length $\epsilon/2$, there is a $t \in I$ such that both $\|E(t + (-\gamma/2, \gamma/2])\xi\|^2$ and $\|E(t + (-\gamma/2, \gamma/2])\eta\|^2$ are smaller than ϵ'. Otherwise

$$\langle E(t + (-\gamma/2, \gamma/2])\xi, \xi\rangle + \langle E(t + (-\gamma/2, \gamma/2])\eta, \eta\rangle \geq \epsilon'$$

for any $t \in I$, which implies that

$$\langle E(J)\xi, \xi\rangle + \langle E(J)\eta, \eta\rangle > 2,$$

for an interval J including I, a contradiction. Thus there is a sequence $(t_1, t_2, \ldots, t_m)$ of points in $\mathbf{T}$ such that $t_{i-1} < t_i$ in the cyclic order and $\epsilon/2 < \mathrm{dist}(t_{i-1}, t_i) < 3\epsilon/2$ with $t_0 = t_m$, and $\|E(J_i)\xi\|^2 < \epsilon'$ and $\|E(J_i)\eta\|^2 < \epsilon'$ for $J_i = t_i + (-\gamma/2, \gamma/2]$.

Let f be a C^∞-function on $\mathbf{T}$ such that $0 \leq f \leq 1$, $\mathrm{supp} f \subset [0, 1/4]$, and $f = 1$ on $[\gamma/2, 1/4 - \gamma/2]$. If ξ, η satisfy the condition in the statement for a sufficiently small δ and for a sufficiently large n, it follows that for any product g of two translates of f,

$$|\langle g(U)\xi, \xi\rangle - \langle g(U)\eta, \eta\rangle| < \epsilon'.$$

We construct a function f_i, as the product of two translates of f, such that $\mathrm{supp} f_i \subset [t_{i-1}, t_i]$ and $f_i = 1$ on $[t_{i-1} + \gamma/2, t_i - \gamma/2]$. Since $|\langle f_i(U)\xi, \xi\rangle - \langle f_i(U)\eta, \eta\rangle| < \epsilon'$, and $\|E(J_i)\xi\|^2 < \epsilon'$ etc., we have that for any i,

$$|\|E(t_{i-1}, t_i)\xi\|^2 - \|E(t_{i-1}, t_i)\eta\|^2| < 2\epsilon'.$$

This completes the proof. $\square$

Proof of Proposition 3.5 By the following lemma, for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ we have a finite subset $\mathcal{G}_A$ of A and $\delta' > 0$ satisfying: Let B be a non-unital C^*-algebra such that A is a unital C^*-subalgebra of the multiplier algebra $M(B)$ of B. If two pure states ω_1, ω_2 of B satisfy that $\omega_1 \sim \omega_2$ and $|\omega_1(x) - \omega_2(x)| < \delta'$, $x \in \mathcal{G}_A$, where ω_i also denotes the natural extension of ω_i to a state on $M(B)$, then there is a continuous path (u_t) in $\mathcal{U}(B)$ such that $u_t - 1 \in B$, $u_0 = 1$, $\omega_1 = \omega_2 \mathrm{Ad}\, u_1$, and

$$\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F}.$$

We may assume, by replacing δ by a smaller one, that $\|x\| \leq 1$ for $x \in \mathcal{G}_A$ and also that $1 \in \mathcal{G}_A$.

Suppose that $B \supset A \otimes C(\mathbf{T})$ with the common unit. Given $\epsilon > 0$ and $\epsilon' = \epsilon^3\delta'/4$ we choose δ, n as in the previous lemma.

Let $\mathcal{G} = \{xz^k \mid x \in \mathcal{G}_A, \ |k| \leq n\}$, where z is the canonical unitary of $C(\mathbf{T}) \simeq 1 \otimes C(\mathbf{T}) \subset A \otimes C(\mathbf{T})$. Let π be an irreducible representation of B and ω_1, ω_2 be the pure states of B defined through unit vectors ξ, η respectively. Suppose that

$$|\omega_1(x) - \omega_2(x)| < \min(\delta, \epsilon'), \quad x \in \mathcal{G}.$$

For $U = \pi(z)$ we have that

$$|\langle U^k \xi, \xi \rangle - \langle U^k \eta, \eta \rangle| < \delta, \quad |k| \leq n.$$

Then we obtain a sequence $(t_1, t_2, \ldots, t_m)$ in $\mathbf{T}$ as in 3.6. Let B_i be the hereditary C^*-subalgebra of B generated by $E(t_{i-1}, t_i)$, where E is the spectral measure of z. There is a natural unital embedding of A into the multiplier algebra $M(B_i)$. Let $f_i \in C(\mathbf{T})$ be a function supported on $[t_{i-1}, t_i]$ as given in the proof of the previous lemma. Then from the proof of 3.6 we may assume that

$$|\omega_1(x f_i(z)) - \omega_2(x f_i(z))| < \epsilon', \quad x \in \mathcal{G}_A,$$

which then implies that

$$|\langle x E(t_{i-1}, t_i)\xi, \xi \rangle - \langle x E(t_{i-1}, t_i)\eta, \eta \rangle| < 2\epsilon', \quad x \in \mathcal{G}_A.$$

We define constants c_i, d_i by

$$d_i \|E(t_{i-1}, t_i)\xi\| = c_i \|E(t_{i-1}, t_i)\eta\| = \min(\|E(t_{i-1}, t_i)\xi\|, \|E(t_{i-1}, t_i)\eta\|),$$

where E is regarded as the spectral measure of $U = \pi(z)$. If

$$d_i \|E(t_{i-1}, t_i)\xi\| = c_i \|E(t_{i-1}, t_i)\eta\| \leq \epsilon^{3/2},$$

then we set $u_{it} = 1 \in \mathcal{U}(B_i)$; otherwise for the states on B_i: $\varphi_1 = \|E(t_{i-1}, t_i)\xi\|^{-2}\omega_1|B_i$ and $\varphi_2 = \|E(t_{i-1}, t_i)\eta\|^{-2}\omega_2|B_i$, it follows that

$$|\varphi_1(x) - \varphi_2(x)| < 4\epsilon'\epsilon^{-3} = \delta', \quad x \in \mathcal{G}_A.$$

Here we have used the fact that $|1 - \|E(t_{i-1}, t_i)\xi\|^{-2}\|E(t_{i-1}, t_i)\eta\|^2| < 2\epsilon'\epsilon^{-3}$ and $\|x\| \leq 1$ for $x \in \mathcal{G}_A$. Hence we obtain a continuous path (u_{it}) in $\mathcal{U}(B_i)$ such that $u_{it} - 1 \in B_i$, $\|\mathrm{Ad}\, u_{it}(x) - x\| < \epsilon$, $x \in \mathcal{F}$, and $\pi(u_{i1}) d_i E(t_{i-1}, t_i)\xi = c_i E(t_{i-1}, t_i)\eta$. Then the sum (u_t) of (u_{it}) over i defines a continuous path in $\mathcal{U}(B)$ (or regarding each u_{it} as in $\mathcal{U}(B)$ take the product of these u_{it} over i). Then, since $\mathrm{dist}(t_{i-1}, t_i) < 3\epsilon/2$, we have that $\|u_t z - z u_t\| < 3\pi\epsilon$ and also that

$$\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F}.$$

Let S be the set of i with $d_i \|E(t_{i-1}, t_i)\xi\| > \epsilon^{3/2}$. Then

$$\sum_{i \notin S} \|d_i E(t_{i-1}, t_i)\xi\|^2 < 2\epsilon^2,$$

and hence

$$\sum_{i \in S} \|d_i E(t_{i-1}, t_i)\xi\|^2 > 1 - 3\epsilon^2,$$

because $\sum_i \|E(\{t_i\})\xi\|^2 < 2\epsilon'\epsilon^{-1} < \epsilon^2\delta'/2$, which can be assumed to be smaller than ϵ^2. Since $\pi(u_1)d_iE(t_{i-1},t_i)\xi = c_iE(t_{i-1},t_i)\eta$, for $i \in S$, we have that

$$\pi(u_1)\sum_{i\in S} d_iE(t_{i-1},t_i)\xi = \sum_{i\in S} c_iE(t_{i-1},t_i)\eta.$$

This implies that

$$\|\pi(u_1)\xi - \eta\| < 2\sqrt{3}\epsilon.$$

This completes the proof. □

Lemma 3.7 *If A is a unital C^*-algebra satisfying Property 2.9. Then for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If B is a non-unital C^*-algebra such that $A \subset M(B)$ with common unit and π is an irreducible representation of B, and ξ, η are unit vectors in $\mathcal{H}_\pi$ such that if*

$$|\langle\pi(x)\xi,\xi\rangle - \langle\pi(x)\eta,\eta\rangle| < \delta, \quad x \in \mathcal{G},$$

where π also denotes the natural extension of π to a representation of $M(B)$, then there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(B)$ such that $u_0 = 1$, $\eta = \pi(u_1)\xi$, $u_t - 1 \in B$, and

$$\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F},\ t \in [0,1].$$

Proof. In the situation as above, for the choice of $(\mathcal{G},\delta)$ made for Property 2.9, we have a continuous path (u_t) in $\mathcal{U}(M(B))$ such that $u_0 = 1$, $\eta \in \mathbf{C}\pi(u_1)\xi$ and

$$\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F},\ t \in [0,1].$$

By multiplying (u_t) by a suitable continuous function, we may assume that $\eta = \pi(u_1)\xi$.

Let (e_ι) be an approximate identity of B. Since $[e_\iota, x] \in B$ converges to zero in the $\sigma(B,B^*)$ topology for any $x \in M(B)$, we can assume, by taking a net from the convex combinations of e_ι's if necessary, that $[e_\iota, x]$ converges to zero in norm for $x \in M(B)$; in particular $[e_\iota, u_t]$ converges to zero as well as $[e_\iota, x]$, $x \in \mathcal{F}$. Thus we find a sequence (e_n) in B such that $0 \le e_n \le 1$, $\pi(e_n)\xi \to \xi$, $[e_n, u_t] \to 0$, $[e_n, x] \to 0$, $x \in \mathcal{F}$, and $e_ne_m - e_m \to 0$ as $n \to \infty$ for each m. Let $\mu \in (0,1)$ be so close to 1 that $\|u_t - u_{\mu t}\| < \epsilon$ for $t \in [0,1]$ and let $k \in \mathbf{N}$ be such that $\|u_{\mu^k t} - 1\| < \epsilon$. For a subsequence (n_i) set

$$z_t = u_te_{n_1} + u_{\mu t}p_1 + u_{\mu^2 t}p_2 + \cdots + u_{\mu^k t}p_k + 1 - e_{n_{k+1}},$$

where $p_i = e_{n_{i+1}} - e_{n_i}$. By assuming that $e_{n_{i+1}}e_{n_i} \approx e_{n_i}$, we have that $p_ip_j \approx 0$ if $|i-j| > 1$. We can also assume that z_t almost commutes with e_{n_i} and $x \in \mathcal{F}$. Hence $z_tp_i \approx u_{\mu^i t}p_i$, $z_te_{n_1} \approx u_te_{n_1}$, and $z_t(1 - e_{n_{k+1}}) \approx 1 - e_{n_{k+1}}$, all up to ϵ. Since $1 = e_{n_1} + \sum_{i=1}^k p_i + 1 - e_{n_{k+1}}$ and $p_iz_t^*z_tp_j$ is arbitrarily close to zero if $|i-j| > 1$ for $i,j = 0,1,\ldots,k+1$ with $p_0 = e_{n_1}$, $p_{k+1} = 1 - e_{n_{k+1}}$, we can conclude that $z_t^*z_t \approx 1$ up to the order of ϵ. (For example $\|z_t^*z_t - 1\| = \|\sum_{i=0}^{k+1}(z_t^*z_t - 1)p_i\| \le \|\sum(z_t^*z_t - 1)p_{2i}\| + \|\sum(z_t^*z_t - 1)p_{2i+1}\| \lesssim$

$2\max_i \|(z_t^*z_t-1)p_i\|$.) Hence z_t is close to a unitary in $B+1$ up to the order of ϵ. By assuming that $\pi(e_{n_1})\xi \approx \xi$, we have that $\pi(z_1)\xi \approx \eta$. In this way we can construct a continuous path (v_t) in $\mathcal{U}(B)$ in a small neighborhood of (z_t) (of order ϵ) such that $\pi(v_1)\xi \approx \eta$, $v_t-1\in B$, and $\|[v_t,x]\|\approx 0$ up to the order of ϵ for $x\in\mathcal{F}$. This completes the proof. $\square$

For $n=1,2,\ldots$ let D_n denote the dimension drop C^*-algebra:

$$D_n=\{x\in C([0,1];M_n)\mid x(0),x(1)\in \mathbf{C}1_n\}.$$

Lemma 3.8 *For any $k,n\in\mathbf{N}$, $D_n\otimes M_k$ has Property 2.9.*

Proof. For any finite subset $\mathcal{F}$ of $D_n\otimes M_k$ and $\epsilon>0$, we find an interval $[a,b]\subset(0,1)$ such that if $x\in\mathcal{F}$, then $\|x(t)-x(0)\|<\epsilon$ for any $t\in[0,a]$ and $\|x(t)-x(1)\|<\epsilon$ for any $t\in[b,1]$. Hence we may suppose that if $x\in\mathcal{F}$, then $x(t)=x(0)\in 1_n\otimes M_k$ for $t\in[0,a]$ and $x(t)=x(1)\in 1_n\otimes M_k$ for $t\in[b,1]$. Replacing a by a smaller one and b by a larger one, we may further suppose that if $x\in\mathcal{F}$, then $x(t)=x(0)$ around $t=a$ and $x(t)=x(1)$ around $t=b$. Denoting by $\mathcal{F}|[a,b]$ the subset of $C[a,b]\otimes M_n\otimes M_k$ obtained by restricting $\mathcal{F}$ to $[a,b]$, we choose $(\mathcal{G}_2,\delta_2)$ for $(\mathcal{F}|[a,b],\epsilon)$ as in 3.7. Similarly we choose $(\mathcal{G}_0,\delta_0)$ for $(\mathcal{F}|[0,a],\epsilon)$ with $\mathcal{F}|[0,a]$ as a subset of $1\otimes M_k$ and $(\mathcal{G}_1,\delta_1)$ for $(\mathcal{F}|[b,1],\epsilon)$ with $\mathcal{F}|[b,1]$ as a subset of $1\otimes M_k$ by using 3.7. We may assume that $\delta_0=\delta_1=\delta_2\equiv\delta$ and $\|x\|\le 1$ for $x\in\mathcal{G}_0\cup\mathcal{G}_1\cup\mathcal{G}_2$. Furthermore we may assume that if $x\in\mathcal{G}_2$, then $x(t)=x(a)$ around $t=a$ in $[a,b]$ and $x(t)=x(b)$ around $t=b$ in $[a,b]$. For $x\in\mathcal{G}_2$, we set

$$\overline{x}(t)=\begin{cases} x(b) & t>b\\ x(t) & a\le t\le b\\ x(a) & t<a\end{cases}$$

and define $\overline{\mathcal{G}}_2=\{\overline{x}\mid x\in\mathcal{G}_2\}\cup\{1\}\subset C[0,1]\otimes M_n\otimes M_k$. If $0<a'<a$ and $b<b'<1$, we can choose $(\overline{\mathcal{G}}|[a',b'],\delta)$ as the $(\mathcal{G},\delta)$ for $(\mathcal{F}|[a',b'],\epsilon)$ in 3.7; because there is an isomorphism of $C[a,b]\otimes M_n\otimes M_k$ onto $C[a',b']\otimes M_n\otimes M_k$ which sends $\mathcal{F}|[a,b]$ onto $\mathcal{F}|[a',b']$ and $\mathcal{G}_2=\overline{\mathcal{G}}_2|[a,b]$ onto $\overline{\mathcal{G}}_2|[a',b']$ respectively.

Let $N\in\mathbf{N}$ be such that $N\delta\epsilon^2>16$, and let $\epsilon'>0$ be such that $N\epsilon'<\min(a,1-b)$. Let

$$I_k=(a-(k+1)\epsilon',a-k\epsilon'],\quad J_k=(b+k\epsilon',b+(k+1)\epsilon']$$

for $k=0,1,\ldots,N-1$. Let $f_k\in C[0,1]$ be such that $0\le f_k\le 1$, $\mathrm{supp} f_k\subset(a-(k+1)\epsilon',b+(k+1)\epsilon']$, and $f_k(t)=1$ for $t\in[a-k\epsilon',b+k\epsilon']$ and let $f_{0k}=(1-f_k)\chi_{[0,a]}\in C[0,1]$ and $f_{1k}=(1-f_k)\chi_{[b,1]}\in C[0,1]$. Finally we define a subset $\mathcal{G}$ of $D_n\otimes M_k$ as the union of $\{f_kx\mid x\in\overline{\mathcal{G}}_2,\ k=0,1,\ldots,N-1\}$, $\{f_{0k}x\mid x\in\mathcal{G}_0 \text{ or } x=1;\ k=0,\ldots,N-1\}$, and $\{f_{1k}x\mid x\in\mathcal{G}_1\text{ or } x=1;\ k=0,\ldots,N-1\}$.

Let B be a C^*-algebra containing $D_n\otimes M_k$ as a unital C^*-subalgebra and let π be an irreducible representation of B. Let ξ,η be unit vectors in $\mathcal{H}_\pi$ such that

$$|\langle\pi(x)\xi,\xi\rangle-\langle\pi(x)\eta,\eta\rangle|<\delta\epsilon^2/4,\quad x\in\mathcal{G}.$$

We first choose k such that

$$s_k \equiv \|\chi_{I_k}\xi\|^2 + \|\chi_{I_k}\eta\|^2 + \|\chi_{J_k}\xi\|^2 + \|\chi_{J_k}\eta\|^2 < \delta\epsilon^2/4,$$

where χ_{I_k} is the characteristic function of I_k regarded as an element of $(D_n \otimes M_k)^{**} \subset B^{**}$, which is also identified with $\pi^{**}(\chi_{I_k})$ etc. (Otherwise $s_k \geq \delta\epsilon^2/4$ for all k leads us to a contradiction because $N\delta\epsilon^2 > 16$.)

Let $a' = a - (k+1)\epsilon'$ and $b' = b + (k+1)\epsilon'$, and let $e = \chi_{(a',b')} \in B^{**}$ and B_2 the hereditary C^*-subalgebra of B corresponding to e. Since $|\langle \pi(f_k x)\xi, \xi\rangle - \langle \pi(f_k x)\eta, \eta\rangle| < \delta\epsilon^2/4, \quad x \in \overline{\mathcal{G}}_2$, we have that

$$|\langle ex\xi, \xi\rangle - \langle ex\eta, \eta\rangle| < \delta\epsilon^2/2, \quad x \in \overline{\mathcal{G}}_2,$$

where we have omitted π and π^{**}. If $\|e\xi\| > \epsilon$ (and ϵ, δ are sufficiently small), then it follows that

$$|\langle \pi_2(x)\xi_2, \xi_2\rangle - \langle \pi_2(x)\eta_2, \eta_2\rangle| < \delta, \quad x \in \overline{\mathcal{G}}_2|[a', b'],$$

where π_2 is the irreducible representation of B_2 on $e\mathcal{H}_\pi$ obtained by restricting π, and $\xi_2 = \|e\xi\|^{-1}e\xi$ and $\eta_2 = \|e\eta\|^{-1}e\eta$. Note that $C[a', b'] \otimes M_n \otimes M_k$ is regarded as a subalgebra of the multiplier algebra of B_2. Hence by 3.7 we can find a continuous path (v_t) in $\mathcal{U}(B_2)$ such that $v_0 = 1$, $v_t - 1 \in B_2$, $\pi_2(v_1)\xi_2 = \eta_2$, and $\|\mathrm{Ad}\, v_t(x) - x\| < \epsilon, \ x \in \mathcal{F}|[a', b']$. If $\|e\xi\| \leq \epsilon$, we set $v_t = 1$. In any case we have that $\pi_2(v_1)\xi_2 \approx \eta_2$ up to ϵ.

Let B_0 be the hereditary C^*-subalgebra of B corresponding to $e_0 = \chi_{[0,a')}$. If $\|e_0\xi\| > \epsilon$, it follows that

$$|\langle \pi_0(x)\xi_0, \xi_0\rangle - \langle \pi_0(x)\eta_0, \eta_0\rangle| < \delta, \quad x \in \mathcal{G}_0,$$

where π_0 is the irreducible representation of B_0 on $e_0\mathcal{H}_\pi$ obtained by restricting π, and $\xi_0 = \|e_0\xi\|^{-1}e_0\xi$ and $\eta_0 = \|e_0\eta\|^{-1}e_0\eta$. Hence we obtain a continuous path (v_{0t}) in $\mathcal{U}(B_0)$ such that $v_{0,0} = 1$, $v_{0t} - 1 \in B_0$, $\pi_0(v_{0,1})\xi_0 = \eta_0$, and $\|\mathrm{Ad}\, v_{0t}(x) - x\| < \epsilon, \ x \in \mathcal{F}|[0, a']$. If $\|e_0\xi\| \leq \epsilon$, then we set $v_{0t} = 1$. As before in any case we have that $\pi_0(v_{0,1})\xi_0 \approx \eta_0$ up to ϵ.

In a similar way we obtain a continuous path (v_{1t}) in $\mathcal{U}(B_1)$ with B_1 the hereditary C^*-subalgebra of B corresponding to $\chi_{(b',1]}$ with similar properties to the above. Since $\|(e + e_0 + e_1)\xi\|^2 = \|e\xi\|^2 + \|e_0\xi\|^2 + \|e_1\xi\|^2 \approx 1$ up to ϵ^2 , we get the desired path in $\mathcal{U}(B)$ by summing these paths $(v_t), (v_{0t})$, and (v_{1t}). □

We recall that A is called an AH algebra (or an approximately homogeneous C^*-algebra) if A is an inductive limit of C^*-algebras of the form $\oplus_{i=1}^k e_i(C_i \otimes M_{n_i})e_i$, where C_i is a unital commutative C^*-algebra and e_i is a projection of $C_i \otimes M_{n_i}$. The class of AH algebras include UHF algebras and AF algebras.

More generally, a C^*-algebra A is called an ASH algebra (or an approximately sub-homogeneous C^*-algebra) if A is an inductive limit of C^*-algebras of the form

$$\bigoplus_{i=1}^{k} e_i(C_i \otimes M_{n_i})e_i \oplus \bigoplus_{j=1}^{r} D_{m_j} \otimes M_{k_j},$$

where the first term is given as above. Some classes of ASH algebras are classified in terms of K theory (see, e.g., [6, 5]); in particular a certain class of ASH algebras of real rank zero is classified by Dadarlat and Gong [5]. The following follows from 3.5, 3.8, and 2.11 (3) and (5).

Theorem 3.9 *Any ASH algebra (in the above sense) has Property 2.9.*

4 Crossed products of AF algebras by Z

Let A be an AF C^*-algebra. If $\alpha \in \overline{\mathrm{Inn}}(A) = \mathrm{AInn}(A)$ has the Rohlin property, then $A \times_\alpha \mathbf{Z}$ is a simple AT algebra and hence, as being an AH algebra, satisfies Property 2.9. More generally we have the following:

Theorem 4.1 *Let A be an AF C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. If the crossed product $A \times_\alpha \mathbf{Z}$ is simple, then $A \times_\alpha \mathbf{Z}$ has Property 2.9.*

Proof. If $A \times_\alpha \mathbf{Z}$ is isomorphic to the compact operators, then this follows from 3.9. Hence we may assume that $A \times_\alpha \mathbf{Z}$ is not of type I. (This assumption will be used only at the end of the proof.)

Since A is an AF C^*-algebra, there is an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A such that $A = \overline{\cup_n A_n}$. By passing to a subsequence of (A_n) we find a sequence (u_n) in $\mathcal{U}(A)$ such that $u_{2n+1} \in A \cap A_{2n}'$ with $A_0 = 0$, $u_{2n} \in A \cap \mathrm{Ad}(u_{2n-1}u_{2n-2}\cdots u_1)\alpha(A_{2n-1})'$, $\|u_n - 1\| < 2^{-n}$, and

$$\mathrm{Ad}\, u_1\alpha(A_1) \subset A_2 \subset \mathrm{Ad}\, u_3u_2u_1\alpha(A_3) \subset A_4 \subset \cdots .$$

By replacing α by $\mathrm{Ad}\, u\alpha$ with $u = \lim_n u_n u_{n-1} \cdots u_1$ and passing to a subsequence of (A_n), we assume that

$$\alpha^{\pm 1}(A_n) \subset A_{n+1}$$

for all n. We denote by 1_n the unit of A_n. It is well-known that the crossed product $A \times_\alpha \mathbf{Z}$ remains the same for this inner perturbation of α.

Let u denote the canonical unitary in $A \times_\alpha \mathbf{Z}$, i.e., $A \times_\alpha \mathbf{Z}$ is generated by A and u with relation that $ux = \alpha(x)u,\ x \in A$.

Let $\mathcal{F}$ be a finite subset of $A \times_\alpha \mathbf{Z}$ and $\epsilon > 0$. By taking a smaller $\epsilon > 0$ if necessary, we can assume that $\mathcal{F}$ is $\{e_{s,ij}^{(k)} \mid i, j, s\} \cup \{1_k u 1_k\}$, where $(e_{s,ij}^{(k)})$ is a family of matrix units for A_k and $s \in \{1, 2, \ldots, N_k\}$ corresponds to each direct summand of A_k.

Assume that $n \in \mathbf{N}$ is so large that $2/n < \epsilon$. We take for $\mathcal{G}$ the set

$$\{1_k e_{s,ij}^{(k+2n)} u^m e_{t,ab}^{(k+2n)} 1_k \mid s, i, j, t, a, b;\ m = 0, \pm 1, \pm 2, \ldots, \pm 2n\},$$

where we assume that 1_k is the sum of some of $e_{t,aa}^{(k+2n)}$. Let B be a C^*-algebra such that $B \supset A \times_\alpha \mathbf{Z}$ and let π be an irreducible representation of B with two unit vectors ξ, η which define pure states ω_1, ω_2 of B respectively. Let $\delta' > 0$ and suppose that

$$|\langle \pi(x)\xi, \xi\rangle - \langle \pi(x)\eta, \eta\rangle| < \delta', \quad x \in \mathcal{G}.$$

We may assume that δ' is so small that either, $\|1_k\xi\| < \epsilon$ and $\|1_k\eta\| < \epsilon$, or otherwise

$$|\langle u^m e_{s,ij}^{(k+2n)}\xi_1, u^\ell e_{t,ab}^{(k+2n)}\xi_1\rangle - \langle u^m e_{s,ij}^{(k+2n)}\eta_1, u^\ell e_{t,ab}^{(k+2n)}\eta_1\rangle| < \delta$$

for $m, \ell = 0, 1, 2, \ldots, n$ and for some prescribed $\delta > 0$, where we have omitted π, and $\xi_1 = \|1_k\xi\|^{-1}1_k\xi$ and $\eta_1 = \|1_k\eta\|^{-1}1_k\eta$.

In the former case we should just take a continuous path (v_t) in $\mathcal{U}((1-1_k)B(1-1_k))$ such that $v_1(1-1_k)\xi \approx (1-1_k)\eta$ and set $u_t = v_t + 1_k$. Since u_t commutes with $\mathcal{F}$ and $u_1\xi \approx \eta$, this completes the proof.

In the latter case suppose that the linear space $\mathcal{L}_{\xi_1}$ spanned by $u^m e_{s,ij}^{k+2n}1_k\xi_1$ with $m = 0, 1, \ldots, n$ and all possible s, i, j is orthogonal to the space $\mathcal{L}_{\eta_1}$ spanned by vectors of the same form with η_1 in place of ξ_1. If δ is sufficiently small, then we obtain vectors $\zeta(m, s, i, j)$ in $\mathcal{H}_\pi \ominus L_{\xi_1}$ for $m = 0, 1, \ldots, n$ and for all s, i, j with $e_{s,i,j}^{(k+2n)}1_k \neq 0$ such that

$$\langle u^m e_{s,ij}^{(k+2n)}\xi_1, u^\ell e_{t,ab}^{(k+2n)}\xi_1\rangle = \langle \zeta(m, s, i, j), \zeta(\ell, t, a, b)\rangle$$

and

$$\|u^m e_{s,ij}^{(k+2n)}1_k\eta - \zeta(m, s, i, j)\| < \epsilon'$$

for a small ϵ'. Then we can define a projection E on $\mathcal{L}_{\xi_1} + \mathcal{L}_\zeta$ such that

$$\begin{aligned} E(u^m e_{s,ij}^{(k+2n)}1_k\xi + \zeta(m, s, i, j)) &= 0, \\ E(u^m e_{s,ij}^{(k+2n)}1_k\xi - \zeta(m, s, i, j)) &= u^m e_{s,ij}^{(k+2n)}1_k\xi - \zeta(m, s, i, j). \end{aligned}$$

Then we find an $h = h^* \in B$ such that $\|h\| = 1$, and $\pi(h)|\mathcal{L}_\xi + \mathcal{L}_\zeta = E$, where ζ denotes $\sum_{s,i}\zeta(0, s, i, i)$ and $\mathcal{L}_\zeta$ is the space spanned by $\zeta(m, s, i, j)$'s. Define

$$\overline{h} = \frac{1}{n}\sum_{j=1}^{n-1}\sum_s\sum_i u^{-j}e_{s,i1}^{(k+n)}he_{s,1i}^{(k+n)}u^j.$$

Then $\|[u, \overline{h}]\| < 2/n < \epsilon$. Since $\sum_s\sum_i e_{s,i1}^{(k+n)}he_{s,1i}^{(k+n)} \in A \cap A'_{k+n}$ and $u^jA_ku^{-j} \subset A_{k+|j|}$, we have that $\overline{h} \in B \cap A'_k$. Also it follows that

$$\begin{aligned} u^{-j}e_{s,i1}^{(k+n)}he_{s,1i}^{(k+n)}u^j(\xi_1 + \zeta) &= u^{-j}e_{s,i1}^{(k+n)}hu^j\alpha^{-j}(e_{s,1i}^{(k+n)})(\xi_1 + \zeta) \\ &\approx u^{-j}e_{s,i1}^{(k+n)}h(\sum c_{s,1i;t,ab}(u^je_{t,ab}^{(k+2n)}\xi_1 + \zeta(j, t, a, b)) = 0, \end{aligned}$$

where $\alpha^{-j}(e_{s,1i}^{(k+n)})1_k = \sum c_{s,1i;t,ab}e_{t,ab}^{(k+2n)}1_k$. Here we have used the fact that $u^je_{t,ab}^{(k+2n)}\zeta \approx u^je_{t,ab}^{(k+2n)}\eta_1 \approx \zeta(j, t, a, b)$. In the same way we obtain that

$$\begin{aligned} u^{-j}e_{s,i1}^{(k+n)}he_{s,1i}^{(k+n)}u^j(\xi_1 - \zeta) &= u^{-j}e_{s,i1}^{(k+n)}hu^j\alpha^{-j}(e_{s,1i}^{(k+n)})(\xi_1 - \zeta) \\ &\approx u^{-j}e_{s,i1}^{(k+n)}(\sum c_{s,1i;t,ab}(u^je_{t,ab}^{(k+2n)}\xi_1 - \zeta(j, t, a, b)) \\ &\approx u^{-j}e_{s,i1}^{(k+n)}u^j\alpha^{-j}(e_{s,1i}^{(k+n)})(\xi_1 - \zeta) \\ &= \alpha^{-j}(e_{s,ii}^{(k+n)})(\xi_1 - \zeta). \end{aligned}$$

Then, since $\alpha^{-j}(1_{k+n}) \geq 1_k$, it follows that

$$\begin{array}{rcl} \overline{h}(\xi_1 + \zeta) & \approx & 0 \\ \overline{h}(\xi_1 - \zeta) & \approx & \xi_1 - \zeta. \end{array}$$

Hence

$$e^{i\pi\overline{h}}\xi_1 = e^{i\pi\overline{h}}(\xi_1 + \zeta)/2 + e^{i\pi\overline{h}}(\xi_1 - \zeta)/2 \approx (\xi_1 + \zeta)/2 - (\xi_1 - \zeta)/2 \approx \zeta.$$

Thus we have that $e^{i\pi\overline{h}}1_k\xi \approx 1_k\eta$. Note that $w_t = e^{i\pi t\overline{h}}$ in $\mathcal{U}(1_kB1_k)$ almost commutes with $\mathcal{F}$. On the other hand we take a continuous path (v_t) in $\mathcal{U}((1-1_k)B(1-1_k))$ such that $v_1(1-1_k)\xi \approx (1-1_k)\eta$. Taking the sum of w_t and v_t completes the proof in the case $\mathcal{L}_\xi \perp \mathcal{L}_\eta$.

If $\mathcal{L}_\xi \not\perp \mathcal{L}_\eta$, then we find a unit vector η' such that $\mathcal{L}_\xi \perp \mathcal{L}_{\eta'}$, $\mathcal{L}_\eta \perp \mathcal{L}_{\eta'}$, and

$$|\langle x\eta, \eta\rangle - \langle x\eta', \eta'\rangle| < \delta', \quad x \in \mathcal{G}$$

for an arbitrarily small $\delta' > 0$. We apply the previous argument to the pairs ξ, η' and η', η to produce appropriate continuous paths $(u_t), (v_t)$ in $\mathcal{U}(B)$; in particular $u_1\xi = \eta'$ and $v_1\eta' = \eta$. Then the product (v_tu_t) satisfies the required properties.

To find such an η' we use the fact that $A \times_\alpha \mathbf{Z}$ is not of type I. Note that the set of vector states of $A \times_\alpha \mathbf{Z}$ in this representation $\rho = \pi|A \times_\alpha \mathbf{Z}$ is weak*-dense in the state space of $A \times_\alpha \mathbf{Z}$. Hence there is a state φ of $A \times_\alpha \mathbf{Z}$ such that π_φ is disjoint from ρ and

$$|\omega_\eta\rho(x) - \varphi(x)| < \delta', \quad x \in \mathcal{G},$$

where we denote by ω_η the state on $\mathcal{B}(\mathcal{H}_\pi)$ defined by the vector η. Then we can find a sequence (ζ_n) of unit vectors in $\mathcal{H}_\pi$ such that $\omega_{\zeta_n}\rho$ converges to φ in the weak* topology. Hence (ζ_n) must converge to zero in the weak topology; we can take the required η' near ζ_n for a sufficiently large n. $\square$

From the proof of the above theorem we obtain:

Theorem 4.2 *Let A be an AF C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. Then the crossed product $A \times_\alpha \mathbf{Z}$ has the transitivity: For any pair of pure states ω_1 and ω_2 of $A \times_\alpha \mathbf{Z}$ with $\ker\pi_{\omega_1} = \ker\pi_{\omega_2}$ there is an $\beta \in \mathrm{AInn}(A \times_\alpha \mathbf{Z})$ such that $\omega_1 = \omega_2\beta$.*

Proof. If $\omega_1 \sim \omega_2$ then this follows from Kadison's transitivity. Thus we may assume that ω_1 and ω_2 are not equivalent. Hence at least the quotient $A \times_\alpha \mathbf{Z}/\ker\pi_{\omega_i}$ does not contain a non-zero type I ideal. We can prove this theorem just as Theorem 2.5 since $A \times_\alpha \mathbf{Z}$ has the following property (where A denotes $A \times_\alpha \mathbf{Z}$). $\square$

Property 4.3 *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: Let B be a C^*-algebra such that $B \supset A$. For any pair of pure states*

ω_1 and ω_2 of B such that $\omega_1 \sim \omega_2$, $B/\ker \pi_{\omega_1}$ does not contain a non-zero type I ideal, and

$$|\omega_1(x) - \omega_2(x)| < \delta, \quad x \in \mathcal{G},$$

there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(B)$ such that $u_0 = 1$, $\omega_1 = \omega_2 \mathrm{Ad}\, u_1$, and

$$\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F}.$$

Lemma 4.4 *Let A be an AF C^*-algebra and $\alpha \in \mathrm{Aut}(A)$. Then the crossed product $A \times_\alpha \mathbf{Z}$ has Property 4.3.*

Proof. The proof of this fact immediately follows from the proof of Theorem 4.1, since the assumption on $B/\ker \pi_{\omega_1}$ substitutes the condition on $A \times_\alpha \mathbf{Z}$ there. □

5 Purely infinite C^*-algebras

Let $\lambda \in (0, 1)$ and let G_λ be the subgroup of $\mathbf{R}$ generated by λ^n with $n \in \mathbf{Z}$. Equipping G_λ with the order coming from $\mathbf{R}$, G_λ, being dense in $\mathbf{R}$, is a dimension group, i.e., there is a stable AF C^*-algebra A_λ whose dimension group is G_λ; A_λ is unique up to isomorphism. Let m_λ denote the automorphism of G_λ defined by the multiplication of λ; there is an automorphism α_λ of A_λ which induces m_λ on G_λ; α_λ is unique up to cocycle conjugacy [7]. If we denote by τ a (densely-defined lower-semicontinuous) trace on A_λ (which is unique up to constant multiple), α_λ satisfies that $\tau\alpha_\lambda = \lambda\tau$. Then it follows from [12] that the crossed product $A_\lambda \times_{\alpha_\lambda} \mathbf{Z}$ is a purely infinite simple C^*-algebra. We can compute $K_*(A_\lambda \times_{\alpha_\lambda} \mathbf{Z})$ by using the Pimsner-Voiculescu exact sequence; $K_0 = G_\lambda/(1-\lambda)G_\lambda$ and $K_1 = 0$. When $\{f \in \mathbf{Z}[t] \mid f(\lambda) = 0\} = p(t)\mathbf{Z}[t]$ with some $p(t) \in \mathbf{Z}[t]$, we have that $K_0 = \mathbf{Z}/p(1)\mathbf{Z}$. By using the classification theory [8], we know that we get all the stable Cuntz algebras $\mathcal{O}_n \otimes \mathcal{K}$ in this way. Recall that $K_0(\mathcal{O}_n) = \mathbf{Z}/(n-1)\mathbf{Z}$ if $n < \infty$ and $K_0(\mathcal{O}_\infty) = \mathbf{Z}$. Hence by using 2.11 and 3.5 we have:

Lemma 5.1 *All the C^*-algebras of the form*

$$C(\mathbf{T}) \otimes M_k \otimes \mathcal{O}_n$$

have Property 2.9, where $n = 2, 3, \ldots, \infty$ with $n = \infty$ inclusive and $k = 1, 2, \ldots, n-1$ if $n < \infty$ and $k = 1, 2, \ldots$ otherwise. Moreover all the C^*-algebras obtained as inductive limits of finite direct sums of C^*-algebras of the above form have Property 2.9.

Note that $K_0(C(\mathbf{T}) \otimes M_k \otimes \mathcal{O}_n) = K_0(\mathcal{O}_n)$ and $K_1(C(\mathbf{T}) \otimes M_k \otimes \mathcal{O}_n) = K_0(\mathcal{O}_n)$. Tensoring with M_k exhausts all possible pairs $(K_0, [1])$ for cyclic groups K_0. In [2] a class of inductive limit C^*-algebras is considered using the C^*-algebras in the above lemma as building blocks. It is not difficult to show that the purely infinite simple separable unital

C^*-algebras obtained this way exhaust all possible (G_0, g, G_1) for $(K_0, [1], K_1)$, where G_0 and G_1 are arbitrary countable abelian groups and $g \in G_0$. (Given any pair of countable abelian groups G_0, G_1 and $g \in G_0$ we find an inductive system (G_{in}, ϕ_{in}) whose limit is G_i, where all G_{in} is a finite direct sum of cyclic groups; for $i = 0$ we specify $g_{0n} \in G_{0n}$ with $\phi_{0n}(g_{0n}) = g_{0,n+1}$ so that (g_{0n}) represents g. Let $G_n = G_{0n} \oplus G_{1n}$ and $g_n = g_{0n} \oplus 0$ and extend $\phi_{in} : G_{in} \to G_{i,n+1}$ to a map $\phi_{in} : G_n \to G_{n+1}$ by adding zero maps. Then it follows that G_i is the inductive limit of (G_n, ϕ_{in}) for $i = 0, 1$. For each G_n we take a direct sum A_n of C^*-algebras of the form $C(\mathbf{T}) \otimes M_k \otimes \mathcal{O}_n$ with $K_*(A_n) = G_n$ and $[1] = g_n$. Then we find a unital homomorphism $\varphi_n : A_n \to A_{n+1}$ such that φ_n induces ϕ_{in} for $i = 0, 1$ and φ_n is a non-zero map from each direct summand of A_n into each direct summand of A_{n+1}; moreover φ_n maps the canonical unitary $z \otimes 1 \otimes 1$ of each direct summand $C(\mathbf{T}) \otimes M_k \otimes \mathcal{O}_n$ of A_n to a non-zero unitary u for each direct summand of A_{n+1} such that the spectrum of u evaluated at each $t \in \mathbf{T}$ is $\mathbf{T}$ (a considerably weaker condition will suffice; see [2] for details); the latter is imposed to insure that the inductive limit A of (A_n, φ_n) is simple. Then A satisfies that $K_i(A) = G_i$ and $[1] = g$.)

Let $\mathcal{C}_\infty$ denote the class of purely infinite separable simple C^*-algebras satisfying the Universal Coefficient Theorem, i.e., $\mathcal{C}_\infty$ is the class of C^*-algebras classified in terms of K theory by Kirchberg and Phillips [8, 10]. Since all of $\mathcal{C}_\infty$ can be obtained as inductive limits as discussed above, we have:

Theorem 5.2 *Any C^*-algebra A in $\mathcal{C}_\infty$ has Property 2.9. In particular* $\mathrm{AInn}(A)$ *acts transitively on $P(A)$.*

6 The group C^*-algebras of discrete amenable groups

Theorem 6.1 *If G is a countable discrete amenable group, then the group C^*-algebra $C^*(G)$ satisfies the transitivity: for any pair of pure states ω_1 and ω_2 with* $\ker \pi_{\omega_1} = \ker \pi_{\omega_2}$ *there is an $\alpha \in \mathrm{AInn}(C^*(G))$ such that $\omega_1 = \omega_2 \alpha$.*

Proof. If $\omega_1 \sim \omega_2$ then this follows from Kadison's transitivity. Thus we may assume that ω_1 and ω_2 are not equivalent. Hence at least the quotient $C^*(G)/I$ with $I = \ker \pi_{\omega_i}$ does not contain a non-zero type I ideal. We can prove this theorem just as Theorem 2.5 (or 4.2) once we have shown the following: □

Lemma 6.2 *If G is a discrete amenable group, $C^*(G)$ has Property 4.3.*

Proof. Since G is a discrete amenable group, for any finite subset $\mathcal{F}$ of G and $\epsilon > 0$ there is a finite subset $\mathcal{G}$ of G such that

$$\frac{|\mathcal{G} \bigtriangleup \mathcal{G}g|}{|\mathcal{G}|} < \epsilon, \quad g \in \mathcal{F},$$

where $\triangle$ denotes the difference of sets and $|\cdot|$ denotes the number of elements. We may suppose that $\mathcal{G} \ni 1$.

Suppose that $C^*(G)$ is a unital C^*-subalgebra of B and let π be an irreducible representation of B. Let $\delta > 0$ and let ξ, η be unit vectors in $\mathcal{H}_\pi$ such that

$$|\langle \pi(g)\xi, \pi(h)\xi\rangle - \langle \pi(g)\eta, \pi(h)\eta\rangle| < \delta$$

for $g, h \in \mathcal{G}$.

First suppose that the linear subspace $\mathcal{L}_\xi$ spanned by $\pi(g)\xi,\ g \in \mathcal{G}$ and the linear subspace $\mathcal{L}_\eta$ spanned by $\pi(g)\eta,\ g \in \mathcal{G}$ are mutually orthogonal. Then for a sufficiently small $\delta > 0$ there is a family $(\zeta(g))$ of vectors in $\mathcal{H} \ominus \mathcal{L}_\xi$, which is infinite-dimensional, such that

$$\|\pi(g)\eta - \zeta(g)\| < \epsilon', \quad g \in \mathcal{G},$$

for small $\epsilon' > 0$ and

$$\langle \pi(g)\xi, \pi(h)\xi\rangle = \langle \zeta(g), \zeta(h)\rangle.$$

Since $\pi(g)\zeta \approx \pi(g)\eta \approx \zeta(g)$ for $g \in \mathcal{G}$ with $\zeta = \zeta(1)$, we have that $\|\pi(g)\zeta - \zeta(g)\| < 2\epsilon'$. Then there is a projection E on $\mathcal{L}_\xi + \mathcal{L}_\zeta$ such that

$$\begin{aligned} E(\pi(g)\xi + \zeta(g)) &= 0, \\ E(\pi(g)\xi - \zeta(g)) &= \pi(g)\xi - \zeta(g). \end{aligned}$$

where $\mathcal{L}_\zeta$ is the subspace spanned by $\zeta(g),\ g \in \mathcal{G}$. Then we find an $h = h^* \in B$ such that $\|h\| = 1$ and

$$\pi(h)|(\mathcal{L}_\xi + \mathcal{L}_\zeta) = E.$$

Set

$$\overline{h} = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} g^{-1}hg,$$

which is a self-adjoint element in B with norm at most one and satisfies that $\|g^{-1}\overline{h}g - \overline{h}\| \leq \epsilon,\ g \in \mathcal{F}$. Note that for $g \in \mathcal{G}$,

$$\|\pi(g^{-1}hg)(\xi + \zeta)\| < 2\epsilon'$$

and

$$\|\pi(g^{-1}hg)(\xi - \zeta) - (\xi - \zeta)\| < 2\epsilon'.$$

Then we obtain that $\pi(e^{i\pi\overline{h}})\xi \approx \zeta$ because

$$\begin{aligned} &\|\pi(e^{i\pi\overline{h}})\xi - \zeta\| \\ &\leq \|\sum_{m=1}^{\infty} \frac{(i\pi)^m}{m!}\pi(\overline{h}^m)\frac{1}{2}(\xi + \zeta)\| + \|\sum_{m=1}^{\infty} \frac{(i\pi)^m}{m!}(\pi(\overline{h}^m) - 1)\frac{1}{2}(\xi - \zeta)\| \\ &\leq \epsilon'(e^\pi - 1 + \pi e^\pi). \end{aligned}$$

Define $u_t = e^{it\pi\overline{h}}$, which is a continuous path in $\mathcal{U}(B)$ satisfying $\|[u_t, g]\| < \pi\epsilon,\ g \in \mathcal{F}$ and $\pi(u_1)\xi \approx \zeta \approx \eta$. Hence this completes the proof in the case that $\mathcal{L}_\xi \perp \mathcal{L}_\eta$.

If $\mathcal{L}_\xi \not\perp \mathcal{L}_\eta$, then we first find a unit vector η' such that $\mathcal{L}_\xi \perp \mathcal{L}_{\eta'}$, $\mathcal{L}_\eta \perp \mathcal{L}_{\eta'}$, and

$$|\langle \pi(g)\eta, \pi(h)\eta\rangle - \langle \pi(g)\eta', \pi(h)\eta'\rangle| < \delta', \quad g, h \in \mathcal{G}$$

for a very small $\delta' > 0$. We apply the previous argument to the pairs ξ, η' and η', η to get the conclusion.

To find such an η' we may argue as in the proof of 2.3 (see also the final part of the proof of 4.1). Let $e \in B$ be such that $0 \leq e \leq 1$ and

$$\|eh^{-1}ge - \omega_2(\pi(h^{-1}g))e^2\| < \delta', \quad g, h \in \mathcal{G},$$

where ω_2 is the vector state defined by η. If there is no such η', then the spectral projection of $\pi(e)$ corresponding to $[1-\delta', 1]$ must be finite-dimensional; i.e., $\pi(B)$ contains the compact operators on $\mathcal{H}_\pi$, which is excluded by the assumption. $\square$

7 Strong transitivity

We first introduce the following property for a C^*-algebra A which is stronger than 2.9:

Property 7.1 *For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If B is a C^*-algebra containing A as a C^*-subalgebra and $\omega_1, \omega_2, \ldots, \omega_n, \varphi_1, \varphi_2, \ldots, \varphi_n$ are pure states of B such that*

1. *(ω_i) are mutually disjoint,*
2. *$\omega_i \sim \varphi_i$ for $i = 1, 2, \ldots, n$,*
3. *$|\omega_i(x) - \varphi_i(x)| < \delta, \quad x \in \mathcal{G},\ i = 1, 2, \ldots, n,$*

then there is a continuous path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(B)$ such that $u_0 = 1$, $\omega_i = \varphi_i \mathrm{Ad}\, u_1,\ i = 1, 2, \ldots, n$, and

$$\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F},\ t \in [0, 1].$$

We can actually show the above property for all the C^*-algebras for which we have shown Property 2.9. This is because we can use the same argument for each pair ω_i, φ_i up to the point where we invoke Kadison's transitivity; instead of working in one irreducible representation of B we are now working in $\bigoplus_{i=1}^n \pi_{\omega_i}$, which is a direct sum of mutually disjoint irreducible representations. We have to find an element h in the C^*-algebra B with the prescribed property in this larger space. But there is a form of Kadison's transitivity in this generality (cf. 1.21.16 of [13]); so we are done.

The following follows just like 2.3.

Lemma 7.2 *If $\omega_1, \omega_2, \dots, \omega_n, \varphi_1, \varphi_2, \dots, \varphi_n$ are pure states of A such that*

1. *(φ_i) are mutually disjoint,*
2. *$\ker \pi_{\omega_i} = \ker \pi_{\varphi_i}$ for $i = 1, 2, \dots, n$,*

then for any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there is a $u \in \mathcal{U}(A)$ such that

$$|\omega_i(x) - \varphi_i \mathrm{Ad}\, u(x)| < \epsilon, \quad x \in \mathcal{F}, \ i = 1, 2, \dots, n.$$

Theorem 7.3 *Suppose that A is a separable C^*-algebra satisfying Property 7.1. Let $(\omega_i)_{1 \leq i \leq n}$ and $(\varphi_i)_{1 \leq i \leq n}$ be finite sequences of pure states of A such that (ω_i) (resp. (φ_i)) are mutually disjoint and $\ker \pi_{\omega_i} = \ker \pi_{\varphi_i}$ for all i. Then there is an $\alpha \in \mathrm{AInn}(A)$ such that $\omega_i = \varphi_i \alpha$ for all $i = 1, 2, \dots, n$. In particular if $(\omega_i)_{1 \leq i \leq n+1}$ are pure states such that they are mutually disjoint maybe except for the pair ω_1, ω_{n+1} and all $\ker \pi_{\omega_i}$'s are equal, then there is an $\alpha \in \mathrm{AInn}(A)$ such that $\omega_i \alpha = \omega_{i+1}$, $i = 1, 2, \dots, n$.*

The proof goes just like the proof of 2.5 does. We present what we use at each induction step in a form of lemma:

Lemma 7.4 *Suppose that A satisfies 7.1. For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If ω_i, φ_i, $i = 1, 2, \dots, n$, are pure states of A such that*

1. *(φ_i) are mutually disjoint,*
2. *$\ker \pi_{\omega_i} = \ker \pi_{\varphi_i}$ for all i,*
3. *$|\omega_i(x) - \varphi_i(x)| < \delta, \quad x \in \mathcal{G}, \ i = 1, 2, \dots, n$,*

then for any finite subset $\mathcal{F}'$ of A and $\epsilon' > 0$ there is a continuous path $(u_t)_{t \in [0,1]}$ in $\mathcal{U}(A)$ such that $u_0 = 1$,

$$\begin{aligned} |\omega_i(x) - \varphi_i \mathrm{Ad}\, u_1(x)| &< \epsilon', \quad x \in \mathcal{F}', \ i = 1, 2, \dots, n, \\ \|\mathrm{Ad}\, u_t(x) - x\| &< \epsilon, \quad x \in \mathcal{F}, \ t \in [0, 1]. \end{aligned}$$

Finally we present another version of transitivity:

Theorem 7.5 *Suppose that A is a separable C^*-algebra with Property 7.1. Let (π_n) and (ρ_n) be sequences of irreducible representations of A such that (π_n) (resp. (ρ_n)) are mutually disjoint and $\ker \pi_n = \ker \rho_n$ for all n. Then there is an $\alpha \in \mathrm{AInn}(A)$ such that $\pi_n = \rho_n \alpha$ for all $n \in \mathbf{N}$. In particular if $(\pi_n)_{n \in \mathbf{Z}}$ are irreducible representations of A such that they are mutually disjoint and all $\ker \pi_n$'s are equal, then there is an $\alpha \in \mathrm{AInn}(A)$ such that $\pi_n \alpha = \pi_{n+1}$ for all n.*

Proof. The proof is similar to the one of 7.3. We will construct pure states ω_n (resp. φ_n) associated with π_n (resp. ρ_n) inductively such that $\omega_n = \varphi_n\alpha$ holds for all n. To introduce a new pair of pure states at each induction step we will use the following easy lemma. $\square$

Lemma 7.6 *Let π and ρ be mutually disjoint irreducible representations of A with $\ker\pi = \ker\rho$ and $u \in \mathcal{U}(A)$. Then for any finite subset $\mathcal{G}$ of A and $\delta > 0$ there are a pure state ω associated with π and a pure state φ associated with ρ such that $|\omega(x) - \varphi\mathrm{Ad}\,u(x)| < \delta,\ x \in \mathcal{G}$.*

Remark 7.7 If A is a non type I separable simple C^*-algebra with Property 7.1, it follows from the above theorem that there is an $\alpha \in \mathrm{AInn}(A)$ such that all non-zero powers α^n are outer. Since for any injective map ω of $\mathbf{Z}$ into the set of equivalence classes of irreducible representations of A there is an $\alpha \in \mathrm{AInn}(A)$ such that $\omega(n)\alpha = \omega(n+1),\ n \in \mathbf{Z}$, it follows that the quotient $\mathrm{AInn}(A)/\mathrm{Inn}(A)$ is uncountable.

8 Remarks on Property 2.9

Looking at the proofs of 3.4–3.8 and 4.1 etc., we come to know that the path (u_t) in Property 2.9 may be chosen so that its length is dominated by a universal constant, which is only slightly bigger than π. (This follows by modifying the proofs given there; it is 2π instead of π that follows immediately.) Taking this fact into consideration, we first introduce the following stronger condition:

Property 8.1 *(for a C^*-algebra A and a constant $C \geq \pi$) For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$ there exist a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: If B is a C^*-algebra containing A as a C^*-subalgebra and ω_1 and ω_2 are pure states of B such that $\omega_1 \sim \omega_2$ and*

$$|\omega_1(x) - \omega_2(x)| < \delta, \quad x \in \mathcal{G},$$

then for any $\epsilon' > 0$ there is a rectifiable path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(B)$ such that $u_0 = 1$, $\omega_1 = \omega_2\mathrm{Ad}\,u_1$, $\mathrm{length}((u_t)) < C + \epsilon'$, and

$$\|\mathrm{Ad}\,u_t(x) - x\| < \epsilon, \quad x \in \mathcal{F},\ t \in [0,1].$$

As asserted above, in the cases we handled in the previous sections, the C^*-algebra A has this property for $C = \pi$. If $A = \mathbf{C}$, then $C = \pi/2$ suffices for the above property to hold. If the property holds for all $A = \mathbf{C}^n$ with $n \in \mathbf{N}$ and for a constant C, then one can check that $C \geq \pi$. To illustrate these points we give two propositions:

Proposition 8.2 *Let $\mathcal{H}$ be a Hilbert space and e be a projection in $\mathcal{H}$. If ξ and η are unit vectors in $\mathcal{H}$ such that $\langle e\xi, \xi\rangle = \langle e\eta, \eta\rangle$, there is a rectifiable path $(u_t)_{t\in[0,1]}$ in the group $\mathcal{U}(\mathcal{H})$ of unitaries on $\mathcal{H}$ such that $u_0 = 1$, $u_1\xi = \lambda\eta$ for some $\lambda \in \mathbf{T}$, $[u_t, e] = 0$, and $\mathrm{length}((u_t)) \leq \pi/2$.*

Proof. There is a $\lambda \in \mathbf{T}$ such that

$$\Re\langle e\xi, \lambda\eta\rangle \geq 0, \quad \Re\langle (1-e)\xi, \lambda\eta\rangle \geq 0.$$

Hence we only have to apply the following proposition to the pairs $e\xi, \lambda e\eta \in e\mathcal{H}$ and $(1-e)\xi, \lambda(1-e)\eta \in (1-e)\mathcal{H}$ separately to reach the conclusion. $\square$

Proposition 8.3 *Let $\mathcal{H}$ be a Hilbert space and ξ, η unit vectors in $\mathcal{H}$. If $\theta \in [0, \pi]$ is such that $\Re\langle \xi, \eta\rangle = \cos\theta$, then there is a rectifiable path (u_t) in $\mathcal{U}(\mathcal{H})$ such that $u_0 = 1$, $u_1\xi = \eta$, and* $\mathrm{length}((u_t)) = \theta$. *Moreover for any rectifiable path (v_t) in $\mathcal{U}(\mathcal{H})$ with $v_0 = 1$ and $v_1\xi = \eta$, it follows that* $\mathrm{length}((v_t)) \geq \theta$.

Proof. If $\eta = e^{\pm i\theta}\xi$, then we set $u_t = e^{\pm it\theta}$.

Suppose that ξ and η are linearly independent. For $t \in [0, 1]$ let

$$\xi(t) = \cos\theta t \cdot \xi + \sin\theta t (\sin\theta)^{-1}(\eta - \cos\theta \cdot \xi).$$

Then $\xi(0) = \xi$ and $\xi(1) = \eta$. Since $\Re\langle \xi, \eta - \cos\theta \cdot \xi\rangle = 0$ and $\|\eta - \cos\theta \cdot \xi\| = \sin\theta$, it follows that $\|\xi(t)\| = 1$ and $\|\xi'(t)\| = \theta$. Thus $(\xi(t))$ is a C^1 path in the unit vectors of $\mathcal{H}$ and its length is θ.

We define $\alpha(t) \in \mathbf{R}$ by

$$i\alpha(t) = \langle \xi'(t), \xi(t)\rangle = i\theta(\sin\theta)^{-1}\Im\langle \eta, \xi\rangle$$

and $\beta(t) \in \mathbf{R}$ by

$$\beta(t) = \|\xi'(t) - i\alpha(t)\xi(t)\| = (\theta - \alpha(t))^{1/2}.$$

Since both $\alpha(t)$ and $\beta(t)$ are constants, we write $\alpha(t) = \alpha$ and $\beta(t) = \beta$. Let $\zeta(t) = \beta^{-1}(\xi'(t) - i\alpha\xi(t))$ and define a self-adjoint operator $h(t)$ by

$$h(t) = \alpha\xi(t) \otimes \xi(t) - i\beta\zeta(t) \otimes \xi(t) + i\beta\xi(t) \otimes \zeta(t) - \alpha\zeta(t) \otimes \zeta(t).$$

Then $(h(t))$ is a continuous path in the set of self-adjoint operators in $\mathcal{H}$ such that $ih(t)\xi(t) = \xi'(t)$ and

$$\|h(t)\| = (\alpha^2 + \beta^2)^{1/2} = \theta = \|\xi'(t)\|.$$

We define a C^1 path $(u(t))_{t\in[0,1]}$ in $\mathcal{U}(\mathcal{H})$ by

$$\frac{d}{dt}u(t) = ih(t)u(t), \quad u(0) = 1.$$

In fact, since $\frac{d}{dt}u(t)^*u(t) = 0 = \frac{d}{dt}u(t)u(t)^*$, we have that $u(t)^*u(t) = 1 = u(t)u(t)^*$. We also have that $\mathrm{length}((u(t)) = \theta$. Since $u(0)\xi = \xi = \xi(0)$ and

$$\begin{aligned}
\frac{d}{dt}\|u(t)\xi - \xi(t)\|^2 &= \frac{d}{dt}(2 - 2\Re\langle u(t)\xi, \xi(t)\rangle) \\
&= -2\Re\langle ih(t)u(t)\xi, \xi(t)\rangle - 2\Re\langle u(t)\xi, \xi'(t)\rangle \\
&= 0,
\end{aligned}$$

we have that $u(t)\xi = \xi(t)$; in particular $u(1)\xi = \eta$. Thus $(u(t))$ is the desired path in $\mathcal{U}(\mathcal{H})$.

Let (v_t) be a rectifiable path in $\mathcal{U}(\mathcal{H})$ such that $v_0 = 1$ and $v_1\xi = \eta$. Since

$$2\cos\theta = 2\Re\langle\xi,\eta\rangle = \langle(v_1^* + v_1)\xi,\xi\rangle,$$

it follows that the spectrum $\mathrm{Spec}(v_1)$ has an $e^{i\varphi}$ such that $\cos\varphi \leq \cos\theta = \Re\langle\xi,\eta\rangle$. For any $\epsilon > 0$ we find a sequence $(t_0, t_1, \ldots, t_n)$ in $[0,1]$ such that $t_0 = 0 < t_1 < t_2 < \cdots < t_n = 1$ and $\|v_{t_i} - v_{t_{i-1}}\| < \epsilon$. Then, by the following lemma, we find a $\lambda_i \in \mathrm{Spec}(v_{t_i})$ such that $\lambda_n = e^{i\varphi}$ and $|\lambda_i - \lambda_{i-1}| \leq \|v_{t_i} - v_{t_{i-1}}\|$. Hence

$$\sum_{i=1}^{n} |\lambda_i - \lambda_{i-1}| \leq \sum_{i=0}^{n} \|v_{t_i} - v_{t_{i-1}}\| \leq \mathrm{length}((v_t)).$$

Since $\lambda_0 = 1$, $\lambda_n = e^{i\varphi}$ and $\epsilon > 0$ is arbitrary, we obtain that $\varphi \leq \mathrm{length}((v_t))$. Since $\theta \leq \varphi$, this completes the proof. □

Lemma 8.4 *Let A be a C^*-algebra and let $u, v \in \mathcal{U}(A)$. If $\lambda \in \mathrm{Spec}(u)$, then there is a $\mu \in \mathrm{Spec}(v)$ such that $|\lambda - \mu| \leq \|u - v\|$.*

Proof. Let $\delta = \|u - v\|$ and $w = u^*v$. Since $\|1 - w\| = \delta$, it follows that $\mathrm{Spec}(w) \subset \{e^{it} \mid |t| \leq \theta\} = \mathbf{T}_\theta$, where $\theta = 2\sin^{-1}\delta/2$. Let ω be a state of A such that $\omega(u) = \lambda \in \mathrm{Spec}(u)$. Then $\omega(w) = \omega(u^*v) = \overline{\lambda}\omega(v)$. Hence $\lambda\omega(w) = \omega(v)$ belongs to the convex closure of $\lambda\mathbf{T}_\theta$, which implies that $\mathrm{Spec}(v) \cap \lambda\mathbf{T}_\theta \neq \emptyset$. □

We seem to need the above stronger property to prove:

Proposition 8.5 *If a C^*-algebra A has Property 8.1 for a constant $C \geq \pi$ and A_1 is a hereditary C^*-subalgebra of A, then A_1 has Property 8.1 for the same constant C.*

To show this we first present the non-unital version of 2.10:

Lemma 8.6 *When A is non-unital, Property 8.1 is equivalent to the one obtained by restricting the ambient C^*-algebra B to a C^*-algebra having an approximate identity for A as an approximate identity for B itself.*

Proof. Technically the proof will be similar to the proofs of 3.7 and 3.8.

We assume the weaker property for A: For $(\mathcal{F}, \epsilon)$ there is a $(\mathcal{G}, \delta)$ satisfying: If $B \supset A$ and $B = \overline{ABA}$, and ω_1 and ω_2 are pure states of B such that $\omega_1 \sim \omega_2$ and $|\omega_1(x) - \omega_2(x)| < \delta$, $x \in \mathcal{G}$, then for any $\epsilon' > 0$ there is a rectifiable path $(u_t)_{t\in[0,1]}$ in $\mathcal{U}(B)$ such that $u_0 = 1$, $\omega_1 = \omega_2 \mathrm{Ad}\, u_1$, $\mathrm{length}((u_t)) < C + \epsilon'$, and $\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon$, $x \in \mathcal{F}$, $t \in [0,1]$.

Let $\mathcal{F}$ be a finite subset of A and $\epsilon > 0$. We may assume that $\|x\| \leq 1$ for $x \in \mathcal{F}$, $\epsilon > 0$ is sufficiently small, and that there is an $e \in A$ such that $0 \leq e \leq 1$ and $exe = x$, $x \in \mathcal{F}$. Let B be a C^*-algebra with $B \supset A$.

Let $\epsilon' > 0$ be such that $2\sqrt{\epsilon'}(C + 8\sqrt{\epsilon'}) < \epsilon$ and let $M, N \in \mathbf{N}$ be so large that $M\epsilon'^2 > 4$ and $N\epsilon' > 2C$. There is a sequence $(e_0, e_1, \dots, e_{M(N+1)})$ in A such that $e_0 x e_0 = x$, $x \in \mathcal{F}$, $0 \le e_i \le 1$, and $e_i e_{i-1} = e_{i-1}$. Let $d > 1$ be a very large constant and let $\mathcal{F}_1 = \mathcal{F} \cup \{de_i \mid 0 \le i \le M(N+1)\}$. For $(\mathcal{F}_1, \epsilon)$ we choose a $(\mathcal{G}, \delta)$ as in the weaker version of Property 8.1. We may assume that $\|x\| \le 1$, $x \in \mathcal{G}$ and $\delta < 1$.

Let π be an irreducible representation of B and ξ, η unit vectors in $\mathcal{H}_\pi$. Suppose that

$$|\langle \pi(x)\xi, \xi\rangle - \langle \pi(x)\eta, \eta\rangle| < \delta\epsilon'^2/2, \quad x \in \mathcal{G}.$$

There exists a j between 1 and $(M-1)(N+1)$ inclusive such that

$$\langle \pi(e_{j+N} - e_{j-1})\xi, \xi\rangle + \langle \pi(e_{j+N} - e_{j-1})\eta, \eta\rangle < \epsilon'^2/2.$$

(Otherwise we would be led to a contradiction, $M\epsilon'^2/2 \le 2$.) If $\langle \pi(e_{j-1})\xi, \xi\rangle > \epsilon'^2$, then we have that for $\xi_1 = \|\pi(e_{j-1}^{1/2})\xi\|^{-1}\pi(e_{j-1}^{1/2})\xi$ and $\eta_1 = \|\pi(e_{j-1}^{1/2})\eta\|^{-1}\pi(e_{j-1}^{1/2})\eta$,

$$|\langle \pi(x)\xi_1, \xi_1\rangle - \langle \pi(x)\eta_1, \eta_1\rangle| < \delta, \quad x \in \mathcal{G}.$$

Here we have used that

$$|1 - \frac{\langle \pi(e_{j-1})\eta, \eta\rangle}{\langle \pi(e_{j-1})\xi, \xi\rangle}| < \delta/2.$$

Then, since $\xi_1, \eta_1 \in \pi(A)\mathcal{H}_\pi$, there is a rectifiable path (u_t) in $\mathcal{U}(\overline{ABA})$ such that $u_0 = 1$, $\pi(u_1)\xi_1 = \eta_1$, length$((u_t)) < C + \epsilon'$, and $\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon$, $x \in \mathcal{F}_1$. Moreover we may assume that the length of $(u_t)_{t\in[0,s]}$ is proportional to s for any $s \in [0,1]$. We set $p_k = e_{j+k} - e_{j+k-1}$ and define

$$z_t = u_t e_j + \sum_{k=1}^{N-1} u_{(1-k/N)t} p_k + 1 - e_{j+N-1}.$$

Let B_0 be the closure of $e_{j+N-1} B e_{j+N-1}$. Then (z_t) is a path in $B_0 + 1$, from which we shall construct a path (v_t) in $\mathcal{U}(B_0)$ with appropriate properties.

Note that $\|[u_t, e_i]\| < \epsilon/d$, where $d \gg 1$ is chosen independently of N; i.e., $\|[u_t, e_i]\| \approx 0$. Since $z_t x = u_t x$ and $x z_t \approx x u_t$ for $x \in \mathcal{F}$, we have that $\|[z_t, x]\| \approx \|[u_t, x]\| < \epsilon$ for $x \in \mathcal{F}$. Thus by making d sufficiently large, we may assume that $\|[z_t, x]\| < \epsilon$, $x \in \mathcal{F}$.

Let $p_0 = e_j$ and $p_N = 1 - e_{j+N-1}$. Then $\sum_{k=0}^N p_k = 1$ and $p_k p_\ell = 0$ if $|k - \ell| > 1$. Let $s(p_k)$ denote the support projection of p_k in B^{**}. Then we have that

$$\begin{aligned}
&\|(z_t - u_{(1-k/N)t}) s(p_k)\|^2 \\
&= \|(u_{(1-(k-1)/N)t} p_{k-1} + u_{(1-k/N)t} p_k + u_{(1-(k+1)/N)t} p_{k+1} - u_{(1-k/N)t}) s(p_k)\|^2 \\
&= \|\{(u_{(1-(k-1)/N)t} - u_{(1-k/N)t}) p_{k-1} + (u_{(1-(k+1)/N)t} - u_{(1-k/N)t}) p_{k+1}\} s(p_k)\|^2 \\
&\approx \|s(p_k)\{p_{k-1}(u_{(1-(k-1)/N)t} - u_{(1-k/N)t})^*(u_{(1-(k-1)/N)t} - u_{(1-k/N)t}) p_{k-1} \\
&\quad + p_{k+1}(u_{(1-(k+1)/N)t} - u_{(1-k/N)t})^*(u_{(1-(k+1)/N)t} - u_{(1-k/N)t}) p_{k+1}\} s(p_k)\| \\
&\le (\frac{C + \epsilon'}{N})^2,
\end{aligned}$$

where $p_{-1} = 0 = p_{N+1}$. Since $C/N < \epsilon'/2$, we may assume that

$$\|(z_t - u_{(1-k/N)t})s(p_k)\| < \epsilon'.$$

Since

$$\begin{aligned} \|z_t^* z_t - 1\| &\approx \|\sum_{k=0}^{N} p_k^{1/2}(z_t^* z_t - 1)p_k^{1/2}\| \\ &= \|\sum_{k=0}^{N} p_k^{1/2}\{(z_t^* - u_{(1-k/N)t}^*)z_t + u_{(1-k/N)t}^*(z_t - u_{(1-k/N)t})\}p_k^{1/2}\| \\ &< \epsilon'(2+\epsilon'), \end{aligned}$$

we may assume that $\|z_t^* z_t - 1\| < 3\epsilon'$. For $0 \le s < t \le 1$, let

$$y(s,t) = \sum_{k=0}^{N} p_k^{1/2}(u_{(1-k/N)t} - u_{(1-k/N)s})^*(u_{(1-k/N)t} - u_{(1-k/N)s})p_k^{1/2}.$$

Then we have that

$$\|y(s,t)\| \le (C+\epsilon')^2(t-s)^2$$

and

$$\begin{aligned} &\|(z_t - z_s)^*(z_t - z_s) - y(s,t)\| \\ &\approx \|\sum p_k^{1/2}(z_t - z_s)^*(z_t - z_s)p_k^{1/2} - y(s,t)\| \\ &= \|\sum p_k^{1/2}\{((z_t - z_s)^* - (u_{(1-k/N)t} - u_{(1-k/N)s})^*)(z_t - z_s) \\ &\quad + (u_{(1-k/N)t} - u_{(1-k/N)s})^*(z_t - z_s - (u_{(1-k/N)t} - u_{(1-k/N)s}))\}p_k^{1/2}\| \\ &\le 2\epsilon'(\|z_t - z_s\| + (C+\epsilon')(t-s)). \end{aligned}$$

Hence we have that

$$\|z_t - z_s\|^2 < (C+\epsilon')^2(t-s)^2 + 2\epsilon'(C+\epsilon')(t-s) + 2\epsilon'\|z_t - z_s\| + \epsilon'',$$

or

$$(\|z_t - z_s\| - \epsilon')^2 < (C+\epsilon')^2(t-s)^2 + 2\epsilon'(C+\epsilon')(t-s) + \epsilon'^2 + \epsilon''$$

for an arbitrarily small constant $\epsilon'' > 0$. Assuming that $t - s > \sqrt{2\epsilon'}$ and $\epsilon'^2 > \epsilon''$, we have that

$$(\|z_t - z_s\| - \epsilon')^2 < (C + \epsilon' + \sqrt{\epsilon'})^2(t-s)^2,$$

which implies that

$$\begin{aligned} \|z_t - z_s\| &< (C + \epsilon' + \sqrt{\epsilon'})(t-s) + \epsilon' \\ &< (C + \epsilon + 2\sqrt{\epsilon'})(t-s) \\ &< (C + 3\sqrt{\epsilon'})(t-s). \end{aligned}$$

We choose a sequence $(t_i)_{i=0}^m$ in $[0,1]$ such that $t_0 = 0 < t_1 < \cdots < t_m = 1$ and

$$\sqrt{2\epsilon'} < t_i - t_{i-1} < 2\sqrt{\epsilon'} < \frac{\epsilon}{C + 8\sqrt{\epsilon'}}.$$

Noting that $\||z_{t_i}| - 1\| < 3\epsilon' \ll \epsilon$, define $v_{t_i} = z_{t_i}|z_{t_i}|^{-1} \in \mathcal{U}(B_0)$. Then we have that for $x \in \mathcal{F}$,

$$\begin{aligned}
\|v_{t_i}x - xv_{t_i}\| &\leq 2\|v_{t_i} - z_{t_i}\| + \|z_{t_i}x - xz_{t_i}\| \\
&\leq 2\|1 - |z_{t_i}|\| + \|[z_{t_i}, x]\| \\
&< 6\epsilon' + \epsilon < 2\epsilon
\end{aligned}$$

and that

$$\begin{aligned}
\|v_{t_i} - v_{t_{i-1}}\| &\leq \|v_{t_i}(1 - |z_{t_i}|)\| + \|z_{t_i} - z_{t_{i-1}}\| + \|v_{t_{i-1}}(1 - |z_{t_{i-1}}|)\| \\
&\leq (C + 3\sqrt{\epsilon'})(t_i - t_{i-1}) + 6\epsilon'.
\end{aligned}$$

Thus, since $5\sqrt{2} > 6$, we have that

$$\|v_{t_i} - v_{t_{i-1}}\| < (C + 8\sqrt{\epsilon'})(t_i - t_{i-1}) < \epsilon.$$

Let

$$\sqrt{-1}h_i = \ln(v_{t_i}v_{t_{i-1}}^*) = \ln(1 - (1 - v_{t_i}v_{t_{i-1}}^*));$$

then

$$\begin{aligned}
\|h_i\| &\leq \|1 - v_{t_i}v_{t_{i-1}}^*\| + \|1 - v_{t_i}v_{t_{i-1}}^*\|^2 \\
&< (C + 8\sqrt{\epsilon'})(t_i - t_{i-1}) + (C + 8\sqrt{\epsilon'})^2 2\sqrt{\epsilon'}(t_i - t_{i-1}) \\
&< (C + (3C^2 + 8)\sqrt{\epsilon'})(t_i - t_{i-1}).
\end{aligned}$$

We define, for $t \in [t_{i-1}, t_i]$,

$$v_t = \exp(\sqrt{-1}(\frac{t - t_{i-1}}{t_i - t_{i-1}})h_i)v_{t_{i-1}}.$$

Then (v_t) is a path in $\mathcal{U}(B_0) \cap (B_0 + 1)$. Moreover (v_t) is a rectifiable path satisfying that

$$\text{length}((v_y)_{y \in [s,t]}) < (C + (3C^2 + 8)\sqrt{\epsilon'})(t - s).$$

If $t \in [t_{i-1}, t_i]$, then $\|v_t - v_{t_{i-1}}\| < \epsilon$ and hence $\|\mathrm{Ad}\, v_t(x) - x\| < 4\epsilon$ for $x \in \mathcal{F}$. Since $\pi(z_1)\xi_1 = \eta_1$ and $\||z_t| - 1\| < 3\epsilon'$, we have that $\|\pi(v_1)\xi_1 - \eta_1\| < 3\epsilon'$. This is how to construct the path (v_t) in $\mathcal{U}(B_0)$ in the case $\langle \pi(e_{j-1})\xi, \xi \rangle > \epsilon'^2$. Otherwise we simply set $v_t = 1 \in \mathcal{U}(B_0)$.

Let B_1 be the closure of $(1 - e_{j+N})B(1 - e_{j+N})$. If $\langle (1 - \pi(e_{j+N}))\xi, \xi \rangle > \epsilon'^2$, set $f = (1 - e_{j+N})^{1/2} \in B_1 + 1$, $\xi_2 = \|\pi(f)\xi\|^{-1}\pi(f)\xi$, and $\eta_2 = \|\pi(f)\eta\|^{-1}\pi(f)\eta$. There is a rectifiable

path (w_t) in $\mathcal{U}(B_1)$ such that $w_1 = 1$, $\pi(w_1)\xi_2 = \eta_2$, and length$((w_u)_{u\in[s,t]}) \le \pi(t-s)$. If $\langle (1-\pi(e_{j+N}))\xi, \xi\rangle \le \epsilon'^2$, set $w_t = 1 \in \mathcal{U}(B_1)$. Let $u_t = (v_t - 1) + (w_t - 1) + 1 \in \mathcal{U}(B)$.

Note that if $\langle \pi(e_{j-1})\xi, \xi\rangle > \epsilon'^2$,

$$|\|\pi(e_{j-1}^{1/2})\xi\| - \|\pi(e_{j-1}^{1/2})\eta\|| < \delta\epsilon'/2 < \epsilon'$$

and that if $\langle \pi(1-e_{j+N})\xi, \xi\rangle > \epsilon'^2$,

$$|\|\pi((1-e_{j+N})^{1/2})\xi\| - \|\pi((1-e_{j+N})^{1/2})\eta\|| < \delta\epsilon'/2 < \epsilon'.$$

Note also that

$$\begin{aligned}
&\|\xi - \pi(e_{j-1}^{1/2})\xi - \pi((1-e_{j+N})^{1/2})\xi\|^2 \\
&= 1 + \langle \pi(e_{j-1})\xi, \xi\rangle + \langle (1-\pi(e_{j+N})\xi, \xi\rangle - 2\langle \pi(e_{j-1}^{1/2})\xi, \xi\rangle - 2\langle \pi((1-e_{j+N})^{1/2})\xi, \xi\rangle \\
&\le \langle \pi(e_{j+N} - e_{j-1})\xi, \xi\rangle < \epsilon'^2/2.
\end{aligned}$$

Thus $\pi(u_1)\xi \approx \eta$ up to the order ϵ'. Since $C \ge \pi$ and $\|u_t - u_s\| = \max(\|v_t - v_s\|, \|w_t - w_s\|)$, it follows that length$((u_t)) < C + (3C^2 + 8)\sqrt{\epsilon'}$. Since w_t commutes with $\mathcal{F}$, it also follows that $\|\mathrm{Ad}\, u_t(x) - x\| < 4\epsilon,\ x \in \mathcal{F}$. We then find a rectifiable path (w_t) in $\mathcal{U}(B)$ such that $w_0 = 1$, $\pi(w_1 u_1)\xi = \eta$, $\|w_t - 1\|$ is of order ϵ', and length$((w_t))$ is of order ϵ', and then form a new path connecting (u_t) with $(w_t u_1)$, which is the desired path. This completes the proof. □

Proof of Proposition 8.5 If the hereditary C^*-subalgebra A_1 has a unit, this follows from (the proof of) 2.11. If A_1 has an approximate identity consisting of projections, this also follows from 2.11. Thus we assume at least that A_1 has no unit.

Let $\mathcal{F}$ be a finite subset of A_1 and $\epsilon > 0$. We may assume that $\|x\| \le 1$ for $x \in \mathcal{F}$, $\epsilon > 0$ is sufficiently small, and that there is an $e \in A_1$ such that $0 \le e \le 1$ and $exe = x,\ x \in \mathcal{F}$. Let B be a C^*-algebra containing A_1. By the previous lemma we may assume that $B = \overline{A_1 B A_1}$.

Let D be a C^*-algebra such that B is a hereditary C^*-subalgebra of D, $D \supset A$, and the following diagram is commutative:

$$\begin{array}{ccc}
A_1 & \subset & B \\
\cap & & \cap \\
A & \subset & D
\end{array}$$

Let $\epsilon' > 0$ be such that $2\sqrt{\epsilon'}(C + 8\sqrt{\epsilon'}) < \epsilon$ and let $M, N \in \mathbf{N}$ be so large that $M\epsilon'^2 > 4$ and $N\epsilon' > 2C$. There is a sequence $(e_0, e_1, \ldots, e_{M(N+1)})$ in A_1 such that $e_0 x e_0 = x,\ x \in \mathcal{F}$, $0 \le e_i \le 1$, and $e_i e_{i-1} = e_{i-1}$. Let d be a very large constant and let $\mathcal{F}_1 = \mathcal{F} \cup \{de_i \mid 0 \le i \le M(N+1)\}$. For $(\mathcal{F}_1, \epsilon)$ with $\mathcal{F}_1$ as a subset of A, we choose $(\mathcal{G}_1, \delta)$ as in Property 8.1 for (A, C). We set

$$\mathcal{G} = \{e_{M(N+1)} x e_{M(N+1)} \mid x \in \mathcal{G}_1\} \cup \{e_i \mid 0 \le i \le M(N+1)\},$$

which is a finite subset of A_1.

Let π be an irreducible representation of B and ξ, η unit vectors in $\mathcal{H}_\pi$. We extend π to an irreducible representation ρ of D; $\mathcal{H}_\pi \subset \mathcal{H}_\rho$. Suppose that

$$|\langle \pi(x)\xi, \xi\rangle - \langle \pi(x)\eta, \eta\rangle| < \delta\epsilon'^2/2, \quad x \in \mathcal{G}.$$

There exists a $j \in \{1, 2, \ldots, (M-1)(N+1)\}$ such that

$$\langle \pi(e_{j+N} - e_{j-1})\xi, \xi\rangle + \langle \pi(e_{j+N} - e_{j-1})\eta, \eta\rangle < \epsilon'^2/2.$$

If $\langle \pi(e_{j-1})\xi, \xi\rangle > \epsilon'^2$, then we have that for the normalized vectors ξ_1, η_1 for $\pi(e_{j-1}^{1/2})\xi$, $\pi(e_{j-1}^{1/2})\eta$ respectively,

$$|\langle \pi(x)\xi_1, \xi_1\rangle - \langle \pi(x)\eta_1, \eta_1\rangle| < \delta, \quad x \in \mathcal{G},$$

which implies that

$$|\langle \rho(x)\xi_1, \xi_1\rangle - \langle \rho(x)\eta_1, \eta_1\rangle| < \delta, \quad x \in \mathcal{G}_1.$$

Then there is a rectifiable path (u_t) in $\mathcal{U}(D)$ such that $u_0 = 1$, $\rho(u_1)\xi_1 = \eta_1$, $\|\mathrm{Ad}\, u_t(x) - x\| < \epsilon$, $x \in \mathcal{F}_1$, and $\mathrm{length}((u_y)_{y\in[s,t]}) < (C + \epsilon')(t - s)$ for $0 \le s < t \le 1$. Let B_0 be the closure of $e_{j+N-1}De_{j+N-1}$, which is a hereditary C^*-subalgebra of B. Then, as in the proof of the previous lemma, we obtain a rectifiable path (v_t) in $\mathcal{U}(B_0)$ such that $v_0 = 1, \pi(v_1)\xi_1 = \eta_1$, $v_t - 1 \in B_0$, $\|\mathrm{Ad}\, v_t(x) - x\| < 4\epsilon$, $x \in \mathcal{F}$, and $\mathrm{length}((v_y)_{y\in[s,t]}) < (C + (3C^2 + 8)\sqrt{\epsilon'})(t - s)$ (by making d sufficiently large). On the other hand if $\langle \pi(e_{j-1})\xi, \xi\rangle \le \epsilon'^2$, we set $v_t = 1 \in \mathcal{U}(B_0)$.

Let B_1 be the closure of $(1 - e_{j+N})B(1 - e_{j+N})$, which is a hereditary C^*-subalgebra of B. If $\langle (1 - \pi(e_{j+N}))\xi, \xi\rangle > \epsilon'^2$, let ξ_2 and η_2 be the normalized vectors for $\pi((1 - e_{j+N})^{1/2})\xi$ and $\pi((1 - e_{j+N})^{1/2})\eta$ respectively. There is a rectifiable path (w_t) in $\mathcal{U}(B_1)$ such that $w_0 = 1$, $\pi(w_1)\xi_2 = \eta_2$, $w_t - 1 \in B_1$, and $\mathrm{length}((w_y)_{y\in[s,t]}) \le \pi(t - s)$. Note that $(w_t - 1)x = 0 = x(w_t - 1)$, $x \in \mathcal{F}$. If $\langle (1 - \pi(e_{j+N}))\xi, \xi\rangle \le \epsilon'^2$, we set $w_t = 1 \in \mathcal{U}(B_1)$. Let $u_t = (v_t - 1) + (w_t - 1) + 1 \in \mathcal{U}(B)$, from which one can construct the desired path as in the proof of the previous lemma. □

Hence Propositions 2.11 and 3.5 can be extended as follows:

Remark 8.7 If $\mathcal{C}_C$ denotes the class of C^*-algebras satisfying Property 8.1 for a constant $C \ge \pi$, then the following statements hold:

1. If A is a non-unital C^*-algebra, $A \in \mathcal{C}_C$ if and only if $\tilde{A} \in \mathcal{C}_C$, where $\tilde{A}$ is the C^*-algebra obtained by adjoining a unit to A.

2. If $A_1, A_2 \in \mathcal{C}_C$, then $A_1 \oplus A_2 \in \mathcal{C}_C$.

3. If $A \in \mathcal{C}_C$ and A_1 is a hereditary C^*-subalgebra of A, then $A_1 \in \mathcal{C}_C$.

4. If $A \in \mathcal{C}_C$ and I is an ideal of A, then $I, A/I \in \mathcal{C}_C$.

5. If (A_n) is an inductive system with $A_n \in \mathcal{C}_C$, then $\lim_n A_n \in \mathcal{C}_C$.

Remark 8.8 If A is a (unital or non-unital) C^*-algebra with Property 8.1 and C is a (unital or non-unital) commutative C^*-algebra, then $A \otimes C$ has Property 8.1 for the same constant.

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